الأفعال السديدة المنتظمة

المستخلص

ان الهدف الرئيسي من هذا العمل هو تقديم نوع جديد (حسب علمنا) من فضاءات $G-$ أسميناه فضاء -G السديد المنتظم .وتضمن البحث بعض الامثلة والمبرهنات المهمة لفضاء -G السديد المنتظم حيث تلك الفضاءات هاوزدورفية.

Regular Proper Actions

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Abstract

The main goal of this work is to create a new type of proper *G* – space , namely, regular proper *G* – space and to explain some of examples and propositions of r - proper action; where X and G is T_2 – spaces.

Introduction

One of the very important concepts in topological groups is the concept of group actions and there are several types of these actions. This paper studies an important class of actions namely, regular proper actions .Proper *G* – spaces were studied by many mathematicians such as group, Bourbaki, Palais, Abels, and others.

Let *B* be a subset of a topological space (*X,T*). We denote the closure of *B* and the interior of *B* by $\overline{B}$ and B^o , respectively.The subset *B* of (*X*, *T*) is called regular open (r – open) if $B = \overline{B}^{\,o}$ .The complement of a regular open set is defined to be a regular closed (r – closed) . If $B = \overline{B^o}$ then the family of all r – open sets in (*X,T*)forms a base of asmaller topology T^ron *X* ,called the semi – regularization of *T* . In section one of this work, we include some of results which then will needed in section two.

In section two, we deal with the definitions, examples, remarks, propositions, theorem and corollaries of regular proper function. Section three recalls the definition of proper *G* – space, gives a new type of proper *G* – space (to the best of our Knowledge), namely, regular proper *G* – space and studies some of its properties, where *G*- space is meant T_2 – space topological *X* on which an r – locally r – compact, non – compact, T_2 – topological group *G* acts continuously on the left.

1. Preliminaries

1.1 Definition [3]:

A subset *B* of a space *X* is called regular open (R – open) set if $B = \overline{B}^{\,o}$. The complement of a regular open set is defined to be regular closed (r – closed) set, then the family of all r – open sets in (*X,T*)forms a base of a smaller topology T^r on *X* ,called the semi – regularization of *T* . In [3] that the subset *B* of *X* is r – open if and only if $B \in T^r$.

1.2 Proposition [3]:
Let X be a space. Then
(i) If A and B are r- open sets then A ∩ B is an r-open set.
(ii)If Ais an r-closed subset of X and B is aclopen set in X, then $A \cap B$ is an r – closed in X.

1.3 Proposition [3]: Let X and Y be two spaces. Then $A_1 \subseteq X$, $A_2 \subseteq Y$ be r – open(r – closed) sets in X and Y , respectively if and only if $A_1 \times A_2$ is r – open(r – closed) in $X \times Y$.

1.4 Definition[3]: A subset B of a space X is called regular neighborhood (r – neighborhood) of $x \in X$ if there is r- open subset O of X such that $x \in O \subseteq B$.

1.5 Definition [3]: Let X and Y be spaces and f: $X \to Y$ be a function. Then:
(i) f is called regular continuous (r – continuous) function if $f^{-1}(A)$ is an r – open set in X for every open set A in Y.
(ii) f is called regular irresolute (r – irresolute) function if $f^{-1}(A)$ is an r – open set in X for every r- open set A in Y.

1.6 Proposition [3]: Let f:$X \to Y$ be a function of spaces. Then f is an r - continuous function if and only if $f^{-1}(A)$ is an r - closed set in X for every closed set A in Y.

1.7 Proposition: Let X and Y be spaces and let f: $X \to Y$ be a continuous, open function. Then f is r – irresolute function.

. **Proof:**

(i)Let A be an r-open set of Y,then A= $\overline{A}^{o}$ .Since f is continues and open then
$f^{-1}(A)$= $f^{-1}(\overline{A}^{o})$=$\left[f^{-1}(\overline{A})\right]^{\circ} = \left[\overline{f^{-1}(A)}\right]^{\circ}$, $f^{-1}(A)$ is an r-open set of X.

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1.8 Definition [3]:
(i) A function f: $X \to Y$ is called regular closed (r – closed) function if the image of each closed subset of X is an r – closed set in Y.
(ii) A function f: $X \to Y$ is called regular open (r – open) function if the image of each open subset of X is an r – open set in Y.

1.9 Definition [3]: Let X and Y be spaces . Then a function f: $X \to Y$ is called a r– homeomorphism if:

(i) f is bijective .

(ii) f is continuous .

(iii) f is r– closed (r– open).

1.10 Proposition[3]: Every r- homeomorphism is homeomorphism.

1.11 Definition [3]: Let $(\chi_d)_{d\in D}$ be a net in a space X, $x\in X$. Then :

i) $(\chi_d)_{d\in D}$ r – converges to x (written $\chi_d \xrightarrow{r} x$) if $(\chi_d)_{d\in D}$ is eventually in every r – neighborhood of x. The point x is called an r – limit point of $(\chi_d)_{d\in D}$, and the notation "$\chi_d \xrightarrow{r} \infty$" is mean that $(\chi_d)_{d\in D}$ has no r – convergent subnet.

ii) $(\chi_d)_{d\in D}$ is said to have x as an r – cluster point [written $\chi_d \overset{r}{\alpha} x$] if $(\chi_d)_{d\in D}$ is frequently in every r - neighborhood of x.

1.12 Proposition: Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) and x_o in X. Then $\chi_d \overset{r}{\alpha} x_o$ if and only if there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{r}_{r} x_o$.

Proof: $\Rightarrow$ Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) such that $\chi_d \overset{\alpha}{} x_o$ in (X, T) . Then $\chi_d \, \alpha \, x_o$ in (X, T^r), so there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \rightarrow x_o$ in (X, T^r). Then $\chi_d \xrightarrow{r} x_o$.

$\Leftarrow$ By same way we will proof only if part.

1.13 Remark: Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) such that $\chi_d{}^{\alpha} x$, $x\in X$ and let A be an open set in X which contains x. Then there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ in A such that $\chi_{dm} \longrightarrow x$.

1.14 Remark [3]: Let X be a space, then:

(i) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \longrightarrow x$ then $\chi_d \xrightarrow{r} x$.

(ii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \; \alpha \; x$ then $\chi_d \overset{r}{\alpha} x$.

(iii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$. Then $\chi_d \xrightarrow{r} x$ in (X, T) if and only if $\chi_d \rightarrow x$ in (X, T^r), and $\chi_d \overset{r}{\alpha} x$ in (X, T) if and only if $\chi_d \, \alpha \, x$ in (X, T^r).

1.15 Remark: Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) such that X is compact T_2- space then $\chi_d{}^{\alpha} x$ if and only if $\chi_d \overset{r}{\alpha} x_o$.

Proof: $\Rightarrow$Clearly,

$\Leftarrow$ Let $(\chi_d)_{d\in D}$ be a net in X such that $\chi_d \overset{r}{\alpha} x$, so by Proposition (1.12) there exists a subnet of $(\chi_d)_{d\in D}$,say itself such that $\chi_d \xrightarrow{r} x$. Since X is compact space then $(\chi_d)_{d\in D}$ has a cluster point ,say y ,then there exists a subnet of $(\chi_d)_{d\in D}$,say itself such that $\chi_d \longrightarrow y$,by remark (1.14) $\chi_d \xrightarrow{r} y$,then x=y (since X is T_2 space then by [3] X is r-T_2 space ,thus χ_d may have unique r-limit point) ,so $\chi_d{}^{\alpha} x$.

1.16 Definition [3]: A subset A of space X is called r – compact set if every r – open cover of A has a finite sub cover. If $A=X$ then X is called an r – compact space.

1.17 Proposition [3]: Let X be a space and F be an r – closed subset of X. Then $F \cap K$ is r – compact subset of F, for every r – compact set K in X.

1.18 Proposition [3]: Let Y be an r – open subspace of space X and $A \subseteq Y$. Then A is an r – compact set in Y if and only if A is an r – compact set in X.

1.19 Definition [3]:

(i) A subset A of space X is called r - relative compact if $\overline{A}$ is r – compact.

(ii) A space X is called r – locally r – compact if every point in X has an r – relative compact r – neighborhood.

1.20 Proposition[3]: Let X and Y be spaces and $f: X \to Y$ be a function, then:
(i)If f is continuous ,then an image f(A)of any compact set A in X is a compact set in Y.
(ii)If f is r-irresolute,then an image f(A) of any r- compact set A in X is an r-compact set in Y.

1.21 Definition [3]: Let $f: X \to Y$ be a function of spaces. Then f is called an regular compact (r-compact) function if $f^{-1}(A)$ is a compact set in X for every r – compact set A in Y.

2 – Regular Proper Function

2.1 Definition [3]: Let X and Y be two spaces. Then $f: X \to Y$ is called regular proper (r - proper) function if :

(i) f is continuous function.

(ii) $f \times I_Z: X \times Z \to Y \times Z$ is r – closed function, for every space Z.

2.2 Proposition [3]: Let X and Y be spaces and $f: X \to Y$ be a continuous function .Then the following statements are equivalent:
(i) f is an r – proper function.

(ii) f is an r – closed function and $f^{-1}(\{y\})$ is a compact set, for each $y \in Y$.

(iii) If $(\chi_d)_{d \in D}$ is a net in X and $y \in Y$ is an r – cluster point of $f(\chi_d)$, then there is a cluster point $x \in X$ of $(\chi_d)_{d \in D}$ such that $f(x) = y$.

2.3 Proposition [3]: Let X, Y and Z be spaces, $f: X \to Y$ and g: $Y \to Z$ be an r – proper functions . Then $g \circ f: X \to Z$ is an r – proper function.
2.4 Proposition [3]; Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be functions. Then $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is an r- proper function if and only if f_1 and f_2 are r – proper functions.

2.5 Proposition [3]:
(i) Every r – proper function is r – closed.
(ii)Every r-proper function is proper..

(iii)Every r-homeomorphism is r-proper.
2.6 Proposition[3]: Let f: $X \to P= \{w\}$ be a function on a space X. Then f is an r – proper function if and only if X is an compact, where w is any point which dose not belongs to X.
. **2.7 Proposition[3]:** Every continuous function from an compact space into a Hausdorff space is r- proper.

2.8 Proposition:[3]: Let X, Y and Z be spaces, f: $X \to Y$ is an r – proper functions and g: $Y \to Z$ is homeomorphism function . Then $g \circ f : X \to Z$ is an r – proper function.

2.9 Proposition [3]: Let X and Y be a spaces, such that Y is a T_2 – space and f: $X \to Y$ be continuous, function. Then the following statements are equivalent:
(i) f is r– compact function.

(ii) f is r– proper function.

3 – Regular Proper G-Space.

3.1 Definition [5]: A topological transformation group is a triple (G,X,φ) where G is a T_2 – topological group, X is a T_2 – topological space and $\varphi : G \times X \to X$ is a continuous function such that:
(i) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ for all $g_1, g_2 \in G$, $x \in X$ and denote g. x *for* $\varphi(g, x)$

(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

3.2 Remark [4]: Let X be a G – space and $x \in X$. Then:
(i) The function φ is called an action of G on X and the space X together with φ is called a G – space (or more precisely left G – space).

(ii) A set $A \subseteq X$ is said to be invariant under G if $GA = A$.

3.3 Definition: A G – space X is called regular proper G – space (r – proper G – space) if the function θ: $G \times X \to X \times X$ which is defined by $\theta(g, x) = (x, g.x)$ is r – proper function.

3.4 Example: The topological group $Z_2 = \{-1, 1\}$ [as Z_2 with discrete topology] acts on the topological space S^n [as a subspace of R^{n+1} with usual topology] as follows:
$\pm 1. (x_1, x_2, \ldots, x_{n+1}) = (\pm x_1, \pm x_2, \ldots, \pm x_{n+1})$

Since Z_2 is an compact, then by Proposition (2.6) the constant function $Z_2 \to P$ is an r – proper. Also the identity function is an r – proper, then by Proposition (2.4) the function of $Z_2 \times S^n$ into $P \times S^n$ is an r – proper.

Since $P\times S^n$ is homeomorphic to S^n, then by Proposition (2.8)the composition $Z_2\times S^n \to S^n$ is an r – proper function . Let φ be the action of Z_2 on S^n. Then φ continuous,. Since S^n is T_2 – space. Then by Proposition (2.7) φ is an r – proper function . Thus by Proposition (2.4) $Z_2\times S^n \to S^n\times S^n$ is an r – proper function ,thus S^n is an r - proper Z_2- space.

3.5 Lemma: If X is a G – space then the function $\theta: G\times X\to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y)\in X\times X$.

Proof: Since: $\theta: G\times X \xrightarrow{I_G\times\Delta} G\times X\times X \xrightarrow{\varphi\times I_X} X\times X \xrightarrow[\cong]{f} X\times X$, where φ is action of G on X. Then $\theta = f o\varphi\times I_X o I_G\times\Delta$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y)\in X\times X$.

3.6 Theorem: Let X be an r – proper G – space and let H be closed subset of G. If Y is an r – open subset of X which is invariant under H, then Y is an r- proper H – space.

Proof: Since X is an r – proper G – space, then the function $\theta: G\times X\to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is an r – proper function. [To prove that $\omega: H\times Y\to Y\times Y$ is an – r – proper function which is defined by $\omega(h, y) = \theta(h, y)$ for each $(h, y)\in H\times Y$.]

(1) Since $\theta: G\times X\to X\times X$ is continuous, then $\omega: H\times Y\to Y\times Y$ is continuous. .

(2) Let $(h_d, y_d)_{d\in D}$ be a net in $H\times Y$ such that $\omega((h_d, y_d)) \overset{r}{\alpha} (x, y)$ for some $(x, y)\in Y\times Y$. Then $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$ in $Y\times Y$. Let A be an r – open subset of $X\times X$ such that $(x, y)\in A$. Since Y is r – open in X, then $Y\times Y$ is an r – open set in $X\times X$. Then $A\cap(Y\times Y)$ is an r – open set in $X\times X$. But $(x, y)\in A\cap(Y\times Y)$ and $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$, thus $(y_d, h_d y_d)$ is frequently in $A\cap(Y\times Y)$ and then $(y_d, h_d y_d)$ is frequently in A, thus $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$ in $X\times X$. since $\theta: G\times X\to X\times X$ is an r – proper function, then by Proposition (2.2) there exists $(h, x_1)\in G\times X$ such that $(h_d, y_d) \alpha (h, x_1)$ and $\theta((h, x_1))=(x, y)$, hence $(x_1, h x_1)=(x,y)$. Thus $x_1=x$ and therefore $h_d \alpha h$. Since $(h_d)_{d\in D}$ is a net in H , and H is r – closed . Then there exists $(h, x)\in H\times Y$ such that $\omega(h, x) = \theta(h,x) = (x, y)$. Then from (1), (2) and by Proposition (2.2) the function $\omega: H\times Y\to Y\times Y$ is an r – proper function. Hence Y is an r – proper H – space.

3.7 Corollary: Let X be an r – proper G – space and Y be an r – open subset of X which is invariant under G. Then Y is an r- proper G – space.

3.8 Corollary: Let X be an r – proper G – space and let H be a closed subset of G. Then X is an r- proper H – space.

3.9 Proposition: Let X be an r – proper G – space, $x \in X$ such that $\{x\}$ is clopen and $T=\{x\}\times X$. Then the function θ_T: $\theta^{-1}(T) \to T$ is an r – proper function, where $\theta: G\times X \to X\times X$ such that θ is an r – irresolute function and $\theta(g, x) = (x, g.x)$, $\forall (g, x) \in G\times X$.

Proof: Since $\{x\}$ is clopen in X then $\{x\}$ is r-clopen set in X. So each $G\times\{x\}$ and $\{x\}\times X$ are r – closed in $G\times X$ and $X\times X$ (respectively). Now, Let F be a closed set in $\theta^{-1}(T) = G\times\{x\}$, then F is a closed in $G\times X$ .Since $F=F\cap(G\times\{x\})$, $\theta_T(F) = \theta(F)\cap(\{x\}\times X)$,since θ *is* r- proper therefore $\theta(F)$ is r– closed in $X\times X$ by Proposition (2.1) $\theta_T(F)$ is r – closed in $X\times X$. But $\theta_T(F) \subseteq \{x\}\times X$, then there exists a subset V of X such that $\theta_T(F) = \{x\}\times V$. Since $\theta_T(F)$ is r- – closed in $X\times X$, so $\{x\}\times V$ is an r – closed set in $\{x\}\times X$, hence $\theta_T(F) = \{x\}\times V$ is an r – closed set in $T= \{x\}\times X$ therefore θ_T: $\theta^{-1}(T) \to T$ is an r – closed. Now, let $(x, y) \in \{x\}\times X$. Since θ is r – proper function, then by Proposition (2.9) θ is an r – compact function. Then $\theta^{-1}(\{(x, y)\})$ is compact in $G\times X$. Then $\theta_T^{-1}(\{(x, y)\})$ is compact set in $G\times\{x\} = \theta^{-1}(T)$. Since θ is continuous, then θ_T: $\theta^{-1}(T) \to T$ is continuous .Thus by Proposition (2.2) θ_T is an r – proper function.

Let X be a G – space and A , B be two subset of X . We mean by $((A, B))$ the set $\{g\in G \,/\, gA\cap B\neq\phi\}$.

From now on, we will use G – space, which satisfies the property if (X,T) and (Y,T') be two spaces and $\forall$ $x_d \longrightarrow x$, $y_d \longrightarrow y$ in X and Y, respectively, then $(x_d\ y_d) \longrightarrow (x,y)$ in product space X×Y .

3.10 Proposition: Let X be a G – space such that X and G are compact space. If for every $x, y\in X$ there exists an r – open set A_x of X contains x and an r – open set A_y of X contains y such that $K=((A_x, A_y))$ is r – relatively compact in G, then X is an r – proper G – space.

Proof: We prove that $\theta: G\times X \to X\times X$, $\theta(g, x) = (x, g.x)$ is an r – proper function. Let $(g_d, \chi_d)_{d\in D}$ be a net in $G\times X$ such that $\theta(g_d, \chi_d) = (\chi_d, g_d\cdot\chi_d) \overset{r}{\alpha} (x, y)$, where $(x, y) \in X\times X$. By proposition (1.15) (x, y) is a cluster point of $\theta(g_d, \chi_d)$. Now, since $x, y\in X$, then there exists an r – open set A_x contains x and an r – open set A_y contains y such that the set $K = ((A_x, A_y))$ is r – relatively compact in G. Thus $A_x\times A_y$ is an open set in $X\times X$ and $(x, y)\in A_x\times A_y$, so by Proposition (1.13) there exists a subnet $(\chi_{d_m}, g_{d_m}\cdot\chi_{d_m})_{d\in D}$ of $(\chi_d, g_d\ \chi_d)$ in $A_x\times A_y$ and

$(\chi_{d_m}, g_{d_m}.\chi_{d_m}) \longrightarrow (x, y)$, hence $\chi_{d_m} \longrightarrow x$ and $g_{d_m}\cdot\chi_{d_m} \longrightarrow y$. Since $\chi_{d_m} \in A_x$, and $g_{d_m}\chi_{d_m} \in A_y$, Then $g_{d_m} . A_x \cap A_y \neq \phi$, $\forall d_m$, so $g_{d_m} \in K$, but K is r – relatively compact in G,then $\overline{K}$ is r-compact in G, since G is compact T_2 space then by [3] $\overline{K}$ is compact in G, then (g_{d_m}) has limit point, say $t \in G$. Since $\chi_{d_m} \longrightarrow x$, then $(g_{d_m}, \chi_{d_m}) \longrightarrow (t,x)$,so θ $((g_{d_m}, \chi_{d_m})) \longrightarrow \theta((t, x))$,i.e. ,$(\chi_{d_m}, g_{d_m}\cdot\chi_{d_m}) \longrightarrow (x, tx)$, thus $g_{d_m}\cdot\chi_{d_m} \longrightarrow tx$ but $g_{d_m}\cdot\chi_{d_m} \longrightarrow y$ and since X is a T_2 space, then $tx = y$. But $(\chi_{d_m}, g_{d_m}.\chi_{d_m})_{d\in D}$ is a subnet of $(\chi_d, g_d\cdot \chi_d)$ and $(g_{d_m}, \chi_{d_m}) \longrightarrow (t, x)$, then $(g_d, \chi_d) \; \alpha \; (t, x)$, thus $\theta((t, x)) = (x,y)$. Thus [by Proposition (2.2)] we have θ is an r – proper function. Hence X is an r – proper G – space.

3.11 Corollary: Let X be a G – space such that G is discrete space. If for every $x, y \in X$ there is an r – open set A_x in X contains x and an r – open set A_y in X contains y such that the set $K = ((A_x, A_y))$ is finite, then X is an r – proper G – space.

Let X be a G – space and $x \in X$. The set $J^r(x)=\{y \in X$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in X with $g_d \xrightarrow{r} \infty$ and $\chi_d \xrightarrow{r} x$ such that $g_d.x \xrightarrow{r} y\}$ is called regular first prolongation limit set of x. $J^r(x)$ is a good tool to discover about the r-proper G – space

3.12 Proposition: Let X be an r-proper G – space then $J^r(x)=\phi$ for each $x \in X$.

Proof: $\Rightarrow$ Suppose that $y \in J^r(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and there is a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{r} x$ such that $g_d\cdot\chi_d \xrightarrow{r} y$, so $\theta((g_d,\chi_d))=(x_d, g_d\cdot\chi_d) \xrightarrow{r} (x, y)$. But X is an r – proper G-space, then by Proposition (2.2) there is $(g, x_1) \in G\times X$ such that $(g_d, x_d) \; \alpha$ (g, x_1). Thus $(g_d)_{d\in D}$ has a subnet (say itself). such that $g_d \longrightarrow g$ then, by Remark 1.14 $g_d \xrightarrow{r} g$, which is contradiction, thus $J^r(x)=\phi$.

3.13 Proposition: Let X be an r – proper G – space with the action $\varphi : G\times X \to X, \varphi(g, x) = g.x, \forall (g,x) \in G\times X$. Then for each $x \in X$, let $x \in X$ such that $\{x\}$ is clopen set in X the function $\varphi_x : G \to X$, which is defined by: $\varphi_x(g)=\varphi(g,x)$ is an r – proper function.

Proof: Let $T=\{x\}\times X \subseteq X\times X$, then by Proposition (3.9) $\theta_T : \theta^{-1}(T) \to T$ is an r – proper function. But:

$\varphi_x =: G \overset{f}{\cong} G\times\{x\} \xrightarrow{\theta_T} \{x\}\times X \overset{h}{\cong} X$, such that f and h are r- homeomorphisms.

Now:

i) since each of these functions are continuous so φ_x: $G\to X$ is continuous.

ii) Let F be a closed in G, then f(F)is a closed in $G\times\{x\}$. Since θ_T: $G\times\{x\}\to \{x\}\times X$ is an r – proper function, then by Proposition (2.5) $\theta_T(f(F))$ is r – closed, then $h(\theta_T(f(F)))$ is an r – closed in X. Then φ_x: $G\to X$ is an r– closed.

iii) Let $y\in X$, then $h^{-1}(\{y\}) = \{(x, y)\}$ such that $x\in X$, since X is T_2 – space, then $\{(x, y)\}$ is an r – closed set in $\{x\}\times X$. Since θ_T is an continuous function, then $\theta_T^{-1}(\{(x, y)\})$ is closed in $G\times\{x\}$, so by Proposition (2.2,ii) $\theta_T^{-1}(h^{-1}(\{y\})) = \theta_T^{-1}\{(x, y)\}$ is compact. Since f is r- homeomorphism, then f^{-1} is a continuous function, so its clear that $f^{-1}(\theta_T^{-1}\{(x, y)\})$ is a compact in G. Then $f^{-1}(\theta_T^{-1}(h^{-1}(\{y\}))) = \varphi_x^{-1}(\{y\})$ is a compact in G. Then by (i),(ii),(iii) and Proposition (2.2,ii) φ_x is an r – proper function.

<u>3.14 Proposition:</u> Let X be an r– proper G – space. then $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N^r(\theta^{-1}(\{(x, y)\}))$, $\exists\, V\in N_r((x, y))$ such that $\theta^{-1}(V)\subseteq U$.

<u>Proof:</u> Since X is an r-proper G – space , then θ: $G\times X\to X\times X$ which is defined by $\theta(g, x) = (x, gx)$, $\forall(g, x)\in G\times X$ is an r- proper function .Let $x, y\in X$ and U be an r – open neighborhood of $\theta^{-1}(x, y)$. Since θ an r – proper function, then by Proposition (2.2.ii) θ is an r – closed function, so $V= (X\times X)\backslash\theta((G\times X)\backslash U)$ is an r – open neighborhood of (x, y) with $\theta^{-1}(V)\subseteq U$. Since θ is continuous and $X\times X$, so by Proposition (2.2) $\theta^{-1}(\{(x, y)\})$ is a compact set $\forall(x, y)\in X\times X$.

<u>3.15 Proposition:</u> Let X be a G – space and θ: $G\times X\to X\times X$ be a function which is defined by $\theta(g, x) = (x, g.x)$, $\forall(g, x)\in G\times X$. Then the following statements are equivalent:

(i) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N_r(((x, y)))$, $\exists V_x\in N_r(x)$ and $V\in N_r(y)$ such that $((V_x, V_y))\subseteq U$.

(ii) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N^r(\theta^{-1}(\{(x, y)\}))$, $\exists\ V\in N_r((x, y))$ such that $\theta^{-1}(V)\subseteq U$.

Proof: i)→ii) Let $x, y\in X$ and let U be an r – neighborhood of $\theta^{-1}(\{(x, y)\})$ = $((x , y))\times\{x\}$. Since $\theta^{-1}(\{(x, y)\})$ is compact, then there are r – neighborhood U' of $((x, y))$ and W of $\{x\}$ such that $U'\times W\subseteq U$, so by (i) there are r – neighborhood V_x of x and V_y of y such that $((V_x, V_y))\subseteq U'$. But $\theta^{-1}((V_x\cap W)\times V_y)\subseteq U'\times W\subseteq U$. Hence (ii), hold.

ii) → i) Let $x, y\in X$ and $U\in N_r((x, y))$. Then $U\times X\in N_r((x, y))\times\{x\}$. Thus $U\times X\in N_r(\theta^{-1}(x, y))$ so by (ii) there exists $V\in N_r(x, y)$ such that $\theta^{-1}(V)\subseteq U\times X$. Then there are r – neighborhood V_x of x and V_y of y such that $\theta^{-1}(V_x\times V_y)\subseteq U\times X$. Hence (i), holds.

3.16 Corollary: Let X be an r – proper G – space , choose a point $x\in X$ and let U be r – neighborhood of the stabilizer G_x of x , then x has an r – neighborhood V such that U contains the stabilizer of all points in V.

Proof: Since U is r – neighborhood of the stabilizer G_x of x, then $U\in N_r(G_x)$. Since G_x = $((x, x))$, then $U\in N_r(((x, x)))$. So by Proposition (3.15)there exist $V_x\in N^r(x, x)$ such that $((V_x, V_x))\subseteq U$. Let $y\in V_x$, then $G_y\subseteq((V_x, V_x))\subseteq U$.

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