ON JACKSON'S THEOREM

Eman Samir Abed-Ali

Department of Mathematics, College of Education, Babylon University,

Hilla, Babil, Iraq.

Abstract

We prove that for a function $f \in W_p^1[-1,1]$, $0 < p < 1$ *and n,r in N(the set of natural numbers) , we have*

$$\left(\int_{-1}^{1} f(x)dx - \sum_{j=1}^{n} \omega_j f(x_j) \right) \le c(r)n^{-1} \int_{0}^{1/n} \frac{\omega_\varphi^{r-1}(f',u)_p}{u^2} du$$

where $-1 < x_1 < \ldots < x_n < 1$ *the roots of Legendre polynomial, and* $\omega_\varphi^m(g,\delta)_p$, *is the Ditzian-Totik mth modulus of smoothness of g in* L_p.

1.Introduction

Let L_p, $0 < p < \infty$ be the set of all functions, which are measurable on $[a,b]$, such that

$$\|f\|_{L_p[a,b]} := \left(\int_a^b |f(x)|^p dx \right)^{1/p} < \infty .$$

And let $W_p^r[a,b]$, be the space of functions that $f^{(r)} \in L_p[a,b]$ and $f^{(r-1)}$ is absolutely continuous in $[a,b]$.

We believe that for approximation in $L_p, p<1$ the measure of smoothness $\omega_\varphi^r(f,\delta)_p$ introduced by Ditzian and Totik [1] is the appropriate tool. Recall that

$$\omega_\varphi^r(f,\delta,[a,b])_p = \sup_{0<h\le\delta}\left(\int_a^b \left|\Delta_{h\varphi(x)}^r(f,x,[a,b])\right|^p dx\right)^{1/p},$$

where

$$\Delta_{h\varphi(x)}^r(f,x,[a,b]) := \begin{cases} \sum_{k=0}^{r} \binom{r}{k}(-1)^{r-k} f\left(x-\frac{rh}{2}+kh\right), & if \quad x\pm\frac{rh}{2}\in[a,b] \\ 0, & o.w. \end{cases}$$

For $[a,b]:=[-1,1]$ for simplicity we write $\|\cdot\|_p = \|\cdot\|_{L_p[a,b]}$, and $\omega_\varphi^r(f,\delta)_p := \omega_\varphi^r(f,\delta,[a,b])_p$.

Recall that the rate of best nth degree polynomial approximation is given by

$$E_n(f)_p := \inf_{p_n\in\Pi_n}\|f-p_n\|_p$$

where Π_n denote the set of all algebraic polynomials of degree not exceeding n.

To prove our theorem we need the following direct result given by:

Theorem 1.1.*[2] For n,r in N and $f\in L_p[-1,1]$*

$$E_n(f)_p \le c\,\omega_\varphi^r\left(f,n^{-1}\right)_p \tag{1}$$

where c is a constant depending on r and p (if $p<1$). For $1 \le p \le \infty$ (1) was proved by Ditzian and Totik [1] and for $0<p<1$, it has been proved by DeVore, Leviatan and Yu [2].

Now, consider the Gaussian Quadrature process [3]

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n} \omega_j f(x_j) =: I_n(f) \qquad (2)$$

based on the roots $-1 < x_1 < \ldots < x_n < 1$ of the nth Legendre polynomial. Since this exact polynomial of degree less than $2n$, we get for the error

$$e_n(f) = \int_{-1}^{1} f(x)dx - I_n(f)$$

in (2) by the definition of the degree of best approximation we have

$$e_n(f) \le 2E_{2n-1}(f)_\infty \qquad (3)$$

where

$$\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$$

(note that $\omega_j \ge 0$ and $\sum_{j=1}^{n} \omega_j$). The crude method of estimating $e_n(f)$ consists of applying Jackson estimate on the right of (3) from (1) we get the sharp inequality

$$e_n(f) \le c\omega_\varphi^r(f, n^{-1})_\infty \qquad (4)$$

which already takes in to account the possibly less smooth behavior of f at ± 1. However the supremum norm in (5) is still too rough, and the natural question is whether for smooth functions one can get upper bounds for $e_n(f)$ using certain $L_p, p<1$ quasi-norm.

R. A. DeVore and L. R. Scott [3] found such estimates, they proved

$$e_n(f) \le c(s) n^{-s} \int_{-1}^{1} \left| f^{(s)}(x) \right| \left(1 - x^2\right)^{5/2} dx \tag{5}$$

first for *s=1* which obviously implies

$$e_n(f) \le c n^{-1} E_{2n-2}(f')_{\varphi, p} \qquad p \ge 1 \tag{6}$$

where $E_n(f)_{\varphi,p}$ means the best weighted approximation with weight $\varphi(x)$ of *f* in L_p defined by

$$E_n(f)_{\varphi,p} := \inf_{p_n \in \Pi_n} \left\| \varphi(f - p_n) \right\|_p .$$

They then proceeded to estimate $E_n(f')_p, p \ge 1,$ using higher derivatives of *f* which finally yielded (5) for any $s \ge 1$.

2. The main result

In this section we introduce our main result. Using (6) we obtain the following theorem

Theorem 2.1. *For* $f \in W_p^1[-1,1], 0 < p < 1$ *we have*

$$e_n(f) \le c(r) n^{-1} \int_0^{1/n} \frac{\omega_\varphi^{r-1}(f', u)_p}{u^2} du \tag{7}$$

Of course the convergence of the integral on the right implies that *f* is L_p equivalent of a locally absolutely continuous function. We use this equivalent representative of *f* in the quadrature formula (Otherwise, we don't have even $e_n(f) = o(1)$)

Proof. Let $p_n \in \Pi_n$ be the best approximating polynomial for *f* in $L_p[-1,1], p<1$. Then $f = p_n + \sum_{k=0}^{\infty} \left(p_{2^{k+1}n} - p_{2^k n}\right)$ in $L_p[-1,1]$ (i.e. the expression in the right is the L_p equivalent of *f* which we need). From (6) and Markov-Bernstein type inequality (see for example [4])

$$
\begin{aligned}
e_n(f) &\le cn^{-1}E_{2n-2}(f')_{q,\varphi} \qquad q\ge 1 \\
&\le cn^{-1}E_n(f')_{q,\varphi} \\
&\le cn^{-1}\left\|\varphi(f'-p_n')\right\|_q \\
&\le cn^{-1}\sum_{k=0}^{\infty} 2^{k+1} n \left\|\varphi\left(p_{2^{k+1}n} - p_{2^k n}\right)\right\|_q .
\end{aligned}
$$

Then using the fact that any two quasi norms are equivalent on the space of algebraic polynomials of a fixed degree we have

$$e_n(f) \le c(p)\sum_{k=0}^{\infty} 2^{k+1} n E_{2^k n}(f)_p \qquad p<1.$$

In view of (1) we get

$$e_n(f) \le c(p)\sum_{k=0}^{\infty} 2^k n\omega_\varphi^r\left(f, 2^{-k} n^{-1}\right)_p .$$

Now since $f \in W_p^1[-1,1], 0<p<1$, so that

$$
\begin{aligned}
e_n(f) &\le c(p) n^{-1}\sum_{k=0}^{\infty} 2^k n\omega_\varphi^r\left(f, 2^{-k} n^{-1}\right)_p \\
&\le c(p) n^{-1}\int_0^{1/n} \frac{\omega_\varphi^{r-1}(f',u)_p}{u^2}\,du .
\end{aligned}
$$

Provided the last integral convergence ♦

As a final remark, we mention that similar bounds holds for many other systems of nodes and in (7) the right hand side has the order

$$\left(\int_{-1}^{x_n}|f|^p+\int_{x_n}^{1}|f|^p\right)^{1/p},$$

for any f constructed from analytic functions, $|x\pm 1|^s$ and iterated logarithms of these, which means that (7) is the best possible estimate for such functions.

References

[1] Z. Ditzian and V. Totik (1987): Moduli of smoothness. Springer-Verlag, Berlin.

[2] R. A. DeVore, D. Leviatan and X. M. Yu (1992): Polynomial approximation in $L_p(0<p<1)$. Constr.approx. 8,187-201.

[3] R. A. DeVore, and L. R. Scott (1984): Error bounds for Gaussian quadrature and weighted L^1 polynomial approximation. SIAM, J. Numer. Anal, 2, 400-412.

[4] K. Kopotun and A. Shadrin (2003): On k-monotone approximation by free knots splines. J. Math. Analysis, 34, 901-924.