r-convergence in metric space

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Abstract. In this paper, firstly we introduce some fundamental concepts are included relating to r-convergence of sequences in a metric space and give some examples. Secondly we consider some differentiations between conventional convergence sequences and r-convergence sequences.

Keywords**:** $r-converge$, $fuzzy\ converge, r-Cauchy,\ fuzzy\ Cauchy, r-bounded$, $fuzzy\ bounded$.

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1. Definitions and Examples

Definition 1.1. Let $r > 0$, a sequence $\{x_n\}$ in a metric space (X,d) is said to be $r-converge$ to $x \in X$, (in symbols $x_n \to_r x$ or $x = r-\lim x_n$) if for every $\varepsilon > 0$, there is an $k \in Z^+$ such that $d(x_n,x) < r+\varepsilon \ \forall\, n > k$.

Remark. The definition of r-convergence implies that $x_n \to_r x$ if, and only if $d(x_n,x) \to r$.

The convergence of the sequence $\{d(x_n,x)\}$ to r takes place in the Euclidean metric space R^1.

Definition 1.2. A sequence $\{x_n\}$ is said to be fuzzy converges to $x \in X$, or that x is an fuzzy limit of $\{x_n\}$; if there is $r > 0$ such that $\{x_n\}$ is an $r-converge$ to x.

Example 1.3. If $X = R$ (the set of real numbers) with usual metric and $\{x_n\} = \{1/n\}$, Then:

1) $x_n \to_1 1$;

2) $x_n \to_1 -1$;

3) $x_n \to_{1/2} 1/2$;

4) $x_n \not\to_{1/2} 1$.

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Ans. (1):

Fix $\varepsilon>0$, from Archimedean property there is $k\in Z^{+}$ such that $(1/k)<\varepsilon$

Let $n>k$, then $(1/n)<(1/k)$.

Now, $d(x_n,1)=\left|1-(1/n)\right|< 1+(1/n) < 1+(1/k) < 1+\varepsilon$.

(2), (3): By the same way.

(4):

Assume that $x_n \to_{1/2} 1$, that is mean for any $\varepsilon>0$ there is $k\in Z^{+}$ such that

$$\left|\frac{1}{n}-1\right|<\frac{1}{2}+\varepsilon \;\; \forall n>k \Rightarrow \left|\frac{1}{n}\right|<\frac{3}{2}+\varepsilon \;\; \forall n>k \Rightarrow -\frac{3}{2}-\varepsilon<\frac{1}{n}<\frac{3}{2}+\varepsilon \;\; \forall n>k .$$

That is mean all but finitely many points are belong to the interval $(-\frac{3}{2}-\varepsilon,\frac{3}{2}+\varepsilon)$.

I.e. the points $x_1,\ldots,x_k$ are out side the interval $(-\frac{3}{2}-\varepsilon,\frac{3}{2}+\varepsilon)$, but this impossible.

Example 1.4. If $X=C$ (the set of complex numbers) with usual metric, $\{z_n\}=\{2+n^{-2}+(2-1/n)i\}$.

Then $z_n \to_1 1+2i$.

Ans. Fix $\varepsilon>0 \Rightarrow \varepsilon/4>0$, from Archimedean property there is $k\in Z^{+}$ such that $(1/k)<\varepsilon/4$.

Since $k\in Z^{+} \Rightarrow 1/k^2<1/k$. Let $n>k \Rightarrow 1/n^2<1/k^2$.

Now,

$$d(z_n,1+2i)^2=\left|z_n-(1+2i)\right|^2=1+3/n^2+1/n^4<1+4/n^2<1+4/k^2<1+4(\varepsilon/4)=1+\varepsilon .$$

Hence, $d(z_n,1+2i)<1+\varepsilon$. Which is complete the proof.

Lemma 1.5. In any metric space (X,d), the r-limit coincides with the conventional limit of a sequence, when $r=0$. (i.e. $x_n \to_0 x$ If, and only if $x_n \to x$).

Proof:

$x_n \to_0 x \Leftrightarrow$ For any $\varepsilon>0$ there is $k\in Z^{+}$ such that $d(x_n,x)<0+\varepsilon=\varepsilon \Leftrightarrow x_n \to x$.

Remark. From the result above, we can say that the concept of an *r*-convergence is a natural extension of the concept of conventional convergence of sequences in metric spaces. However, there is some properties in conventional convergence of sequences are not satisfies in *r*-convergence, as shown in example (1.3.) that $x_n \to_1 1$ and $x_n \to_1 -1$, but $1 \neq -1$. That is mean the 1-convergence of a sequence $\{1/n\}$ is not unique. Conversely there is some properties are not satisfies in the conventional convergence, but it is true in r-convergence sequences (cf. Theorem 2.6, Theorem 2.9, Corollary 2.10).

Lemma 1.6. If (X,d) be a metric space and $\{x_n\}$ is $r-converge$ to x, then $\{x_n\}$ is $q-converge$ to x for any $q > r$.

Proof:

$\{x_n\}$ is $r-converge$ to x, then for every $\varepsilon > 0$, there is an $k \in Z^+$ such that $d(x_n, x) < r + \varepsilon \quad \forall\, n > k$, but $r + \varepsilon < q + \varepsilon$, the result is clarity.

Lemma 1.7. Let $\{x_n\},\{y_n\},\{z_n\}$ are sequences in a metric space (X,d) and $\{x_n\}$ is the disjoint union of $\{y_n\},\{z_n\}$, then $\{x_n\}$ is an $r-converge$ to $x \in X$ if, and only if both $\{y_n\},\{z_n\}$ are an $r-converge$ to x.

Proof:

($\Rightarrow$) Assume that $\{x_n\}$ is an $r-converge$ to x, then for any $\varepsilon > 0$, there is $k \in Z^+$ such that $d(x_n, x) < r + \varepsilon \ \ \forall\, n > k$, but any element in a sequence $\{x_n\}$ is either in $\{y_n\}$ or $\{z_n\}$ not in both. Hence, $d(y_n, x) < r + \varepsilon \ \ and\, d(z_n, x) < r + \varepsilon \quad \forall n > k$, we get the result.

($\Leftarrow$) If $\{y_n\},\{z_n\}$ are an $r-converge$ to x. Then for any $\varepsilon > 0$, there are $k_1, k_2 \in Z^+$ such that $d(y_n, x) < r + \varepsilon \ \ \forall n > k_1 \ and\, d(z_n, x) < r + \varepsilon \quad \forall n > k_2$, now the assumption $\{y_n\} \cup \{z_n\} = \{x_n\}$ and $\{y_n\} \cap \{z_n\} = \phi$ implies that $d(x_n, x) < r + \varepsilon \ \ \forall n > k = \max\{k_1, k_2\}$, and the proof is complete.

Example 1.8. In R with usual metric let us consider sequences:

$\{x_n\} = \{1 + (1/n)\}$, $\{y_n\} = \{1 + (-1)^n\}$ and $\{z_n\} = \{1 + [(1-n)/n]^n\}$. A sequence $\{x_n\}$ has the conventional limit equal to 1 and many fuzzy limits (e.g., $x_n \to_1 0, 0.5, 2$). Sequence $\{y_n\}$ does not have the conventional limit but has different fuzzy limits (e.g., $y_n \to_1 0$, but $y_n \to_2 1, -1, 0.5$). Sequence $\{z_n\}$ does not have the conventional limit but has a variety of fuzzy limits (e.g., $z_n \to_1 1$ and $z_n \to_2 0, 0.5, 1.5, 1.7, 2$).

Thus, we see that many sequences that do not have the conventional limit but have lots of fuzzy limits.

2. MAIN RESULTS

Theorem 2.1. If (X, d) be a linear metric space. $\{x_n\}, \{y_n\}$ are sequences in X such that $x_n \to_r x$ and $y_n \to_q y$ then:

a) $x_n + y_n \to_{r+q} x + y$ where $x_n + y_n = \{x_n + y_n ; n \in Z^+\}$;

b) $x_n - y_n \to_{r+q} x - y$ where $x_n - y_n = \{x_n - y_n ; n \in Z^+\}$;

c) $\beta x_n \to_{|\beta| \cdot r} \beta x$ for any $\beta \in R$ where $\beta x_n = \{\beta \cdot x_n ; n \in Z^+\}$.

Proof:

To prove part (c):

$x_n \to_r x$ implies to for any $\varepsilon_1 > 0$ there is $k \in Z^+$ such that $d(x_n, x) < r + \varepsilon_1 \quad \forall n > k$

Now, $\beta \cdot d(x_n, x) < |\beta| \cdot d(x_n, x) < |\beta| \cdot r + |\beta| \cdot \varepsilon_1 = |\beta| \cdot r + \varepsilon \quad \forall n > k$, when $\varepsilon = |\beta| \cdot \varepsilon_1$.

Theorem 2.2. Let (X, d) be a metric space , $x_n \to_r x$ and $y_n \to_q y$ then $\{d(x_n, y_n)\} \to_{r+q} d(x, y)$.

Proof:

Assume that $x_n \to_r x$ and $y_n \to_q y$, then :

$\forall \varepsilon > 0, \exists k_1, k_2 \in Z^+ \ni d(x_n, x) < r + \varepsilon/2 \quad \forall n > k_1$ and $d(y_n, y) < q + \varepsilon/2 \quad \forall n > k_2$

Let $k = \max\{k_1, k_2\}$

$$\begin{aligned} | d(x_n, y_n) - d(x, y) | &= | d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x, y) | \\ &\le | d(x_n, y_n) - d(x_n, y) | + | d(x_n, y) - d(x, y) | \\ &\le d(y_n, y) + d(x_n, x) < q + \varepsilon/2 + r + \varepsilon/2 = (r+q) + \varepsilon. \end{aligned}$$

Thus, $\{d(x_n, y_n)\} \to_{r+q} d(x, y)$.

Definition 2.3. A sequence $\{x_n\}$ in a metric space (X, d) is called :

(1) $r-Cauchy$ if for any $\varepsilon > 0$ there is $k \in Z^+$ such that for every $n > m > k$ we have $d(x_n, x_m) < 2r + \varepsilon$.

(2) Fuzzy Cauchy, if there is $r > 0$ such that it is $r-Cauchy$.

Lemma 2.4. A sequence $\{x_n\}$ is a conventional Cauchy if, and only if it is $0-Cauchy$.

Proof:

$\{x_n\}$ is $0-Cauchy \Leftrightarrow$ for any $\varepsilon>0$ there is $k\in Z^+$ such that for every $n>m>k$ we have $d(x_n,x_m)<2(0)+\varepsilon=\varepsilon \Leftrightarrow \{x_n\}$ is a conventional Cauchy.

Lemma 2.5. Any $r-Cauchy$ sequence is $q-Cauchy$ for all $q>r$.

Proof:

Assume that $\{x_n\}$ is $r-Cauchy$ sequence, then:

For any $\varepsilon>0$ there is $k\in Z^+$ such that $d(x_n,x_m)<2r+\varepsilon$, but $2r+\varepsilon<2q+\varepsilon$.

Theorem 2.6. A sequence $\{x_n\}$ in a metric space (X,d) is fuzzy converges if, and only if it is fuzzy Cauchy.

Proof:

$(\Rightarrow)$ If a sequence $\{x_n\}$ is fuzzy converges , then there is $r>0$ such that $\{x_n\}$ is an $r-converge$ to $r-$limit, say x.(i.e. $\forall\ \varepsilon>0$, $\exists\, k\in Z^+ \ni d(x_n,x)<r+\varepsilon/2\ \ \forall n>k$).

Now, if $n,m\geq k$ then:

$d(x_n,x_m)\leq d(x_n,x)+d(x_m,x)<2r+2(\varepsilon/2)=2r+\varepsilon$

Thus, $\{x_n\}$ is fuzzy Cauchy.

$(\Leftarrow)$ Assume that $\{x_n\}$ is fuzzy Cauchy, then there is $r>0$ such that $\{x_n\}$ is $r-Cauchy$.

That is mean for any $\varepsilon>0$ there is $k\in Z^+$ such that $d(x_n,x_m)<2r+\varepsilon\ \ \forall n,m>k$.

Take $\alpha=k+1$, $q=2r$, then $\alpha\in Z^+$ and $q>0$. We get $d(x_n,x_\alpha)<q+\varepsilon\ \ \forall n>\alpha$.

By other words a sequence $\{x_n\}$ is $q-converge$ to x_α. Thus, $\{x_n\}$ is fuzzy converges.

The proof is complete.

Definition 2.7. Let (X,d) be a metric space. If x is a point of X and $r>0$, then for any $\varepsilon>0$:

(1) The $r-open\ ball$ (in symbols $B_\varepsilon^r(x)$) with center x and radius r is the subset of X defined by $B_\varepsilon^r(x)=\{\,y\in X:\ d(x,y)<r+\varepsilon\,\}$.

(2) The $r-closed\ ball$(in symbols $B_\varepsilon^r[x]$) with center x and radius r is the subset of X defined by $B_\varepsilon^r[x]=\{\,y\in X:\ d(x,y)\leq r+\varepsilon\,\}$.

Definition 2.8. A sequence $\{x_n\}$ in a metric space (X,d) is said to be:

(1) $r-bounded$ if there is $x \in X$ such that $\{x_n\} \subset B_\varepsilon^r(x)$ for any $\varepsilon > 0$.

(2) fuzzy bounded if there is $r > 0$ such that it is $r-bounded$.

Theorem 2.9. A sequence $\{x_n\}$ in a metric space (X,d) is fuzzy converges if, and only if it is fuzzy bounded.

Proof:

$(\Rightarrow)$ Assume that a sequence $\{x_n\}$ is fuzzy converges, then there are $x \in X$ and $r > 0$ such that $\{x_n\}$ is $r-converge$ to x. By other word, for any $\varepsilon > 0$, there is $k \in Z^+$ such that $d(x_n, x) < r + \varepsilon \ \forall n > k$.

Take $\alpha > \max\{r, d(x_1,x),...,d(x_k,x)\}$, then $\alpha \in Z^+$ and $d(x_n,x) < r + \varepsilon < \alpha + \varepsilon$.

Thus, $\{x_n\} \subset B_\varepsilon^\alpha(x) \Rightarrow \{x_n\}$ is $\alpha - bounded$ or $\{x_n\}$ is fuzzy bounded.

$(\Leftarrow)$ If $\{x_n\}$ is fuzzy bounded, then there are $x \in X$, $r > 0$ such that $\{x_n\} \subset B_\varepsilon^r(x) \ \forall n$.

By other word $d(x_n, x) < r + \varepsilon \ \forall n$. Thus, $\{x_n\}$ is fuzzy converges.

The proof is complete.

Corollary 2.10. A sequence $\{x_n\}$ in a metric space (X,d) is fuzzy Cauchy if, and only if it is fuzzy bounded.

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