Generalized Derivations with Commutativity of Prime Rings

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Abstract

The main purpose of this paper is to study and investigate some results concerning generalized derivations (D,d) and (G,g) of prime ring R ,when the additive mapping acts as a left centralizer of R ,we obtain either d=0 or g=0,and when R contain a non –zero ideal U,R is commutative.

Key words: *Generalized derivations, centralizer,commutativity and prime rings.*

AMS Mathematics Subject Classification (2000) : 16W25, 16N60, 19U80.

1.Introduction and Preliminaries

Generalized derivation of operators on various algebraic structures have been an active area of research since the last fifty years due to their usefulness in various fields of mathematics . Throughout R will represent an associative ring with the center Z(R), A ring R is 2-torsion free in case 2x = o implies that x = o for any x∈R. Recall that R is prime if xRy=o implies x=o or y=o ,and R is semiprime if xRx=o implies x=o. A prime ring is semiprime but the converse is not true in general . An additive mapping d:R→R is called a derivation if d(xy)=d(x)y+xd(y) holds for all x,y ∈ R and d is called left centralizer if d(xy)=d(x)y for all x,y∈ R. A mapping d is called centralizing if [d(x),x]∈Z(R) for all x∈R, in particular , if [d(x),x] = o for all x∈R , then it is called commuting , and is called central if d(x)∈Z(R) for all x∈R . Every central mapping is obviously commuting but not conversely in general . In [1] ,Bresar defined the following notation ,an additive mapping D:R→R is said to be a generalized derivation if there exists a derivation d:R→R such that D(xy) =D(x)y+xd(y)for all x,y∈ R.Hence the concept of a generalized derivation covers both the concepts of a derivation and of a left multiplier (i.e.an additive map d satisfying d(xy) =d(x)y for all x,y ∈ R,[2]).Other properties of generalized derivations were given by B.Hvala [3] ,T.K.Lee [4] and A.Nakajima ([5] ,[6] and[7]) .We note that for a semiprime ring R,if D is a function from R *to R and d:R→R is an additive mapping such that D(xy)=D(x)y+xd(y) for all x,y∈ R . Then D uniquely determined by d and moreover d must be a derivation by [[1], Remark1] .We denote a generalized derivation D:R→R determined by a derivation d of R by (D,d) .We write [x,y]=xy-yx and note that important identity [xy,z]=x[y,z]+[x,z]y and [x,yz]=y[x,z]+[x,y]z.And the symbol xoy stands for the anti- commutator xy + yx . Some authors have studied centralizers in the general framework of semiprime rings (see[8-12]) . Muhammad A .C. and Mohammed S.S.[13] proved , let R be a semiprime ring and d : R → R a mapping satisfy d(x)y = xd(y) for all x,y∈R . Then d is a centralizer . Molnar [14] has proved , let R be a 2-torsion free prime ring and let d : R → R be an additive mapping . If d(xyx) = d(x)yx holds for every x,y∈R, then d is a left centralizer . Muhammad A.C. and A. B. Thaheem [15] proved , let d and g be a pair of derivations of semiprime ring R satisfying d(x)x + xg(x)∈Z(R) , then cd and cg are central for all c∈Z(R). A.B. Thaheem [16] has proved , if d and g is a pair of derivations on semiprime ring R satisfying d(x)x + xg(x) = o for all x∈R, then d(x),g(x)∈Z(R) and d(u)[x,y] = g(u)[x,y] = o for all u , x,y∈R .J. Vukman [17] proved , let R be a 2-torsion free semiprime ring and let d : R → R be an additive centralizing mapping on R , in this case , d is commuting on R .B. Zalar [12] has proved , let R be a 2-torsion free semiprime ring and d : R → R an additive mapping which satisfies* $d(x^2) = d(x)x$ *for all x∈R . Then d is a left centralizer. Mohammad A.,Asma A . and Shakir A.[18] proved , let R be a prime ring and U be a non-zero ideal of R . If R admits a generalized derivation D associated with a non-zero derivation d such that D(xy) - xy∈Z(R) for all x,y∈U , then R is commutative . Hvala [3] initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation . Nadeem[19] proved , let R be a prime ring and U a non-zero ideal of R . If R admits a generalized derivation D with d such that D(xoy) = xoy holds for all x,y∈U , and if D = o or d≠o , then R is commutative , where d is derivation and the symbol xoy stands for the anti- commutator xy + yx. Recently Asharf [20] has investigates the commutativity of a prime ring R admitting a generalized derivation D with associated derivation d satisfying [d(x),D(y)] = o for all x,y∈U, where U is a non-zero ideal of R. . In this paper we study study and investigate some results concerning generalized derivations (D,d) and (G,g) of prime ring R ,when the additive mapping acts as a left centralizer of R ,we obtain either d=0 or g=0, and when R contain a non-zero ideal , R is commutative .*

To achieve our purposes ,we mention the following results .

Lemma 1[21:Lemma1]

The center of semiprime ring contains no non-zero nilpotent elements. *In[22] proved, Let R be a semiprime ring and let* $a\in R$ *.If* $a^2=o$ *then* $a\in Z(R)$*.In our paper we shall extend this result to the following lemma.*

Lemma 2

Let R be a semiprime ring.If $x^2=o$ *then* $x\in Z(R)$ *for all* $x\in R$ *.*

Proof: *For any* $x,y\in R$*,we get* $xy+yx=0$ *for all* $x,y\in R$*.*
Replacing y by yz and using the fact that $xy=-yx$*,we find that*
$y[x,z]=0$ *for all* $x,y,z\in R$*.Replacing y by* $[x,z]r$ *for all* $x,r,z\in R$
and using the semiprimeness of R,we obtain $[x,z]=0$ *for all* $x,z\in R$ **.**
Thus,we get $x\in Z(R)$ *for all* $x\in R$ *.*

Lemma 3[23 :Main Theorem]

Let R be a semiprime ring , d a non-zero derivation of R , and U a non-zero left ideal of R .If for some positive integers $t_o,t_1,\ldots,t_n$ *and all* $x\in U$ *, the identity* $[[\ldots[[d(x^{to}),x^{t1},x^{t2}],\ldots],x^{tn}]=o$ *holds , then either* $d(U)=o$ *or else* $d(U)$ *and* $d(R)U$ *are contained in non-zero central ideal of R . In particular when R is a prime ring , R is commutative***.**

2.The Main Results

Theorem 2.1

Let R be a prime ring,(D,d) and (G,g) be generalized derivations of R,if R admists to satisfy $[d(x),g(x)]=o$ *for all* $x\in R$ *and d acts as a left centralizer (resp .g acts as a left centralizer) ,then either* $d(x)=0$ *or* $g(x)=0$*.*

Proof: *We have* $[d(x),g(x)]=o$ *for all* $x\in R$*.Replacing x by xy ,we obtain* $[d(x)y,g(xy)]+[xd(y),g(xy)]=o$ *for all* $x,y\in R$*.*
$d(x)[y,g(xy)]+[d(x),g(xy)]y+x[d(y),g(xy)]+[x,g(xy)]d(y)=o$ *for all* $x,y\in R$*.Then*
$d(x)[y,g(x)y]+d(x)[y,xg(y)]+[d(x),g(x)y]y+[d(x),xg(y)]y+x[d(y),g(x)y]+x[d(y),xg(y)]+[x,g(x)y]d(y)+[x,xg(y)]d(y)=o$ *for all* $x,y\in R$*.*
$d(x)[y,g(x)]y+d(x)x[y,g(y)]+d(x)[y,x]g(y)+g(x)[d(x),y]y+[d(x),g(x)]y^2+x[d(x),g(y)]y+[d(x),x]g(y)y+xg(x)[d(y),y]+x[d(y),g(x)]y+x^2[d(y),g(y)]+x[d(y),x]g(y)+g(x)[x,y]d(y)+[x,g(x)]yd(y)+x[x,g(y)]d(y)=o$ *for all* $x,y\in R$*.*
Replacing y by x and according to the relation $[d(x),g(x)]=o$*,we obtain* $d(x)[x,g(x)]x+d(x)x[x,g(x)]+g(x)[d(x),x]x+[d(x),x]g(x)x+xg(x)[d(x),x]+x[d(x),x]g(x)+[x,g(x)]xd(x)+x[x,g(x)]d(x)=o$ *for all* $x\in R$*.Then*
$d(x)xg(x)x-d(x)g(x)x^2+d(x)x^2g(x)-d(x)xg(x)x+g(x)d(x)x^2-g(x)xd(x)x+d(x)xg(x)x-xd(x)g(x)x+xg(x)d(x)x-xg(x)xd(x)+xd(x)xg(x)-x^2d(x)g(x)+xg(x)xd(x)-g(x)x^2d(x)+x^2g(x)d(x)-xg(x)xd(x)=o$ *for all* $x\in R$*. Then*
$d(x)x^2g(x)-g(x)xd(x)x+d(x)xg(x)x-xd(x)g(x)x+xg(x)d(x)x-$

$xg(x)xd(x)+xd(x)xg(x)-x^2d(x)g(x)-g(x)x^2d(x)+x^2g(x)d(x)=o$ for all $x\in R$.Since $d(x)g(x)=g(x)d(x)$,then above equation become
$d(x)x^2g(x)-g(x)xd(x)x+d(x)xg(x)x-xg(x)xd(x)+xd(x)xg(x)-g(x)x^2d(x)=o$ for all $x\in R$.

Since d is acts as a left centralizer ,then
$d(x^3)g(x)-g(x)xd(x^2)+d(x^2)g(x)x-xg(x)xd(x)+xd(x^2)g(x)-g(x)x^2d(x)=o$ for all $x\in R$. Then
$d(x)x^2g(x)+xd(x^2)g(x)-g(x)xd(x)x-g(x)x^2d(x)$
$+d(x)xg(x)x+xd(x)g(x)x-xg(x)xd(x)+xd(x)xg(x)+x^2d(x)g(x)$
$-g(x)x^2d(x)=o$ for all $x\in R$. According to (5) , we obtain
$xd(x^2)g(x)-g(x)x^2d(x)+xd(x)g(x)x+x^2d(x)g(x)=o$
for all $x\in R$.
then $xd(x)xg(x)+2x^2d(x)g(x)-g(x)x^2d(x)+xd(x)g(x)x=o$
for all $x\in R$.
Since $d(x)g(x)=g(x)d(x)$, above equation become
$xd(x)xg(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x)x=o$
for all $x\in R$. (1)
Since d acts as a left centralizer , (1) become
$xd(x^2)g(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x^2)=o$ for all $x\in R$. Then
$xd(x)xg(x)+x^2d(x)g(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x)x+$
$xg(x)xd(x)=o$ for all $x\in R$. (2)
According to (1) the equation (2) become
$x^2g(x)d(x)+xg(x)xd(x)=o$ for all $x\in R$.
Since $g(x)d(x)=d(x)g(x)$, we obtain
$x^2d(x)g(x)+xg(x)xd(x)=o$ for all $x\in R$. (3)
Since d acts as a left centralizer , (3) become
$x^2d(xg(x))+xg(x)xd(x)=o$ for all $x\in R$. Then
$x^2d(x)g(x)+x^3d(g(x))+xg(x)xd(x)=o$ for all $x\in R$. (4)
According to (3), the equation (4) reduce to
$x^3d(g(x))=o$ for all $x\in R$. (5)
Right- multiplying (5) by r and since d acts as a left centralizer , we get
$x^3d(g(x))r+x^3g(x)d(r)=o$ for all $x,r\in R$.
According to (5) , we obtain
$x^3g(x)d(r)=o$ for all $x,r\in R$. (6)
Left –multiplying (6)by $x^2g(x)d(x)x$ and right –multiplying by x^2 with using Lemmas (1 and 2) , we obtain
$x^2g(x)d(r)x^2=o$ for all $x,r\in R$. (7)
Left – multiplying (7) by $g(x)d(r)$ with using Lemma (1and 2) , we get $g(x)d(r)x^2=o$ for all $x,r\in R$. (8)

Right – multiplying (8) by $g(x)d(r)x$ and left- multiplying by x with using Lemmas (1 and 2), we obtain
$xg(x)d(r)x=o$ for all $x,r\in R$. By using same technique in (8) ,we obtain $g(x)d(r)x=o$ for all $x,r\in R$. (9)
Since d acts as a left centralizer , we get
$g(x)d(r)x+g(x)rd(x)=o$ for all $x,r\in R$. (10)
According to (9) , the equation (10) reduces to $g(x)Rd(x)=o$.
Then by primeness of R,we obtain
either $d(x)=0$ or $g(x)=0$.

Corollary 2.2

Let R be a prime ring and U anon-zero ideal,(D,d) and (G,g) be generalized derivations of R,if R admists to satisfy $[d(x),g(x)]=o$ for all $x\in R$ and d acts as a left centralizer (resp .g acts as a left centralizer) ,then R is commutative.

Proof:*By using same techniques in Theorem2.1,we can get either $d(x)=0$ or $g(x)=0$.*
If $d(x)=0$,then left-multiplying by x,gives $d(x)x=0$ for all $x\in R$.Again right – multiplying by x,gives $xd(x)=0$ for all $x\in R$.By subtracting theis relations,we obtain $[d(x),x]=0$ for all $x\in R$.Again by same techniques,we can get $[[d(x),x],x]=0$ for all $x\in R$.Then by Lemma3,we obtain R is commutative. Similarly for $g(x)=0$.

Theorem 2.3

Let R be a prime ring,(D,d) and (G,g) be generalized derivations of R, if R a dmits to satisfy $[D(x),G(x)]=o$ for all $x \in R$ and d acts as a left centralizer (resp.g acts as a left centralizer).Then either $d(x)=0$ or $g(x)=0$.

Proof:*We have $[D(x),G(x)]=o$ for all $x\in R$. Replacing x by xy ,we obtain $[D(x)y,G(xy)]+[xd(y),G(xy)]=o$ for all $x,y\in R$. Then $D(x)[y,G(xy)]+[D(x),G(xy)]y+x[d(y),G(xy)]+[x,G(xy)]d(y)=o$ for all $x,y\in R$.*
$D(x)[y,G(x)y]+D(x)[y,xg(y)]+[D(x),G(x)y]y+[D(x),xg(y)]y+x[d(y),G(x)y]+x[d(y),xg(y)]+[x,G(x)y]d(y)+[x,xg(y)]d(y)=o$ for all $x,y\in R$. Then
$D(x)[y,G(x)]y+D(x)x[y,g(y)]+D(x)[y,x]g(y)+G(x)[D(x),y]y+x[D(x),g(y)]y+[D(x),x]g(y)y+xG(x)[d(y),y]+x[d(y),G(x)]y+x^2[d(y),g(y)]+x[d(y),x]g(y)+G(x)[x,y]d(y)+[x,G(x)]yd(y)+x[x,g(y)]d(y)=o$ for all $x,y\in R$.Replacing y by x ,we obtain
$D(x)[x,G(x)]x+D(x)x[x,g(x)]+G(x)[D(x),x]x+x[D(x),g(x)]x+[D(x),x]g(x)x+xG(x)[d(x),x]+x[d(x),G(x)]x+x^2[d(x),g(x)]+x[d(x),x]g(x)+[x,G(x)]xd(x)+x[x,g(x)]d(x)=o$ for all $x\in R$.
$x([D(x),g(x)]x+G(x)[d(x),x]+[d(x),G(x)]x+x[d(x),g(x)]+[d(x),x]g(x)+[x,g(x)]d(x))+D(x)[x,G(x)]x+D(x)x[x,g(x)]+G(x)[D(x),x]x+[D(x),x]g(x)x+[x,G(x)]xd(x)=o$ for all $x\in R$. Then
$x(D(x)g(x)x-g(x)D(x)x+G(x)d(x)x-G(x)xd(x)+d(x)G(x)x-G(x)d(x)x+xd(x)g(x)-$

$xg(x)d(x)+d(x)xg(x)-xd(x)g(x)+xg(x)d(x)-g(x)xd(x))+D(x)xG(x)-D(x)G(x)x+D(x)x^2g(x)-D(x)xg(x)x+G(x)D(x)x^2-G(x)xD(x)x+D(x)xg(x)x-xD(x)g(x)x+xG(x)xd(x)-G(x)x^2d(x)=o$ *for all* $x\in R$. *Then*

$-xg(x)D(x)x+xd(x)G(x)x+xd(x)xg(x)-xg(x)xd(x)+D(x)xG(x)x+D(x)x^2g(x)-G(x)xD(x)x-G(x)x^2d(x)=o$ *for all* $x\in R$.

Since d acts as a left centralizer ,then

$-xg(x)D(x)x+xd(x)G(x)x+xd(x^2)g(x)-xg(x)xd(x)+D(x)xG(x)x+D(x)x^2g(x)-G(x)xD(x)x-G(x)x^2d(x)=o$ *for all* $x\in R$. (11)

$-xg(x)D(x)x+xd(x)G(x)x+xd(x)xg(x)+x^2d(x)g(x)-xg(x)xd(x)+D(x)xG(x)x+D(x)x^2g(x)-G(x)xD(x)x-G(x)x^2d(x)=o$ *for all* $x\in R$. (12)

Since d acts as a left centralizer (12) reduce to

$-xg(x)D(x)x+xd(x)G(x)x+xd(x^2)g(x)+x^2d(x)g(x)-xg(x)xd(x)+D(x)xG(x)x+D(x)x^2g(x)-G(x)xD(x)x-G(x)x^2d(x)=o$ *for all* $x\in R$. *According to (11) ,we obtain* $x^2d(x)g(x)=o$ *for all* $x\in R$. *Left –multiplying by* $xd(x)g(x)$ *and right –multiplying by x with using Lemmas (1 and 2),we get*

$xd(x)g(x)x=o$ *for all* $x\in R$. *Right –multiplying by* $d(x)g(x)$*with using Lemmas (1 and 2),we obtain* $xd(x)g(x)=o$ *for all* $x\in R$. *Left –multiplying by* $d(r)$ *we get*

$d(r)xd(x)g(x)=o$ *for all* $x\in R$. (13)

Since d acts as a left centralizer ,we obtain $d(rx)d(x)g(x)=o$ *for all* $x\in R$. *Then* $d(r)xd(x)g(x)+rd(x)d(x)g(x)=o$ *for all* $x\in R$. *According to (13) ,above reduces to*

$rd(x)d(x)g(x)=o$ *for all* $x\in R$. (14)

Left –multiplying (14) by $d(x)^2g(x)$*, we obtain*

$d(x)^2g(x)=o$ *for all* $x\in R$. (15)

Right –multiplying (15)by $d(x)$ *and left –multiplying by* $d(x)g(x)$*with using Lemmas (1 and 2),we get*

$d(x)g(x)d(x)=o$ *for all* $x\in R$. (16)

Left –multiplying (16) by $g(x)$*with using Lemmas (1 and 2),we get* $d(x)g(x)=o$ *for all* $x\in R$. (17)

Right –multiplying (17)by $rd(x)$ *and left –multiplying by* $g(x)r$ *with using Lemmas (1 and 2) ,we obtain* $g(x)Rd(x)=o$ *. Then by primeness of R,we obtain either* $d(x)=0$ *or* $g(x)=0$.

By the same techniques in Corollary 2.2 ,we can prove the following corollary.

Corollary2.4

Let R be a prime ring and U a non-zero ideal,(D,d) and (G,g) be generalized derivations of R, if R a dmits to satisfy $[D(x),G(x)]=o$ *for all* $x\in R$ *and d acts as a left centralizer (resp.g acts as a left centralizer), then R is commutative..*

Theorem 2.5

Let R be a prime ring,(D,d)and (G,g)be generalized derivations of R,if R admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x∈ R and d acts as a left centralizer(resp.g acts as a left centralizer) .Then either d(x)=0 or g(x)=0.

Proof: *We have [D(x),G(x)]=[d(x),g(x)] for all x∈ R. Then D(x)G(x)-G(x)D(x)=d(x)g(x)-g(x)d(x) for all x∈ R. Since d acts as a left centralizer ,we obtain D(x)G(x)-G(x)D(x)=d(xg(x))-g(x)d(x) for all x∈ R. Then*

[D(x),G(x)]=d(x)g(x)-xd(g(x))-g(x)d(x) for all x∈ R. According to the relation [D(x),G(x)]=[d(x),g(x)] ,we get

-xd(g(x))=o for all x∈ R. (18)

Right –multiplying (18)by y and since d acts as a left centralizer, we obtain –xd(g(x)y)=o for all x,y∈R. Then

-xd(g(x))y-xg(x)d(y)=o for all x,y∈R. According to (18),we obtain xg(x)d(y)=o for all x,y∈R. (19)

Left –multiplying (19) by d(r) ,we obtain d(r)xg(x)d(y)=o for all x,y,r∈ R. Since d acts as a left centralizer ,we get d(rx)g(x)d(y)=o for all x,y,r∈ R. Then according to (19) ,we obtain rd(x)g(x)d(y)=o for all x,y,r∈ R.

Replacing r by g(x) and y by x with using Lemmas (1 and 2) we get

g(x)d(x)=o for all x∈R. Left-multiplying by d(x)r and right –multiplying by rg(x)with using Lemmas(1 and 2),we obtain d(x)Rg(x)=o . Then by primeness of R,we obtain either d=0 or g=0.

By the same techniques in Corollary 2.2 ,we can prove the following corollary.

Corollary 2.6

Let R be a prime ring and U a non-zero ideal ,(D,d)and (G,g)be generalized derivations of R,if R admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x∈ R and d acts as a left centralizer(resp.g acts as a left centralizer)), then R is commutative

Theorem 2.7

Let R be a prime ring,(D,d) and (G,g) be generalized derivations of R, if R admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x∈R and d and g acts as a left centralizers.Then either d=0 or g=0.

Proof: *We have [D(x),G(x)]=[d(x),g(x)] for all x∈R. Then [D(x),G(x)]=d(x)g(x)-g(x)d(x) for all x∈R. Since d and g acts as a left centralizer , we get [D(x),G(x)]=d(xg(x))-g(xd(x)) for all x∈R. Then [D(x),G(x)]=d(x)g(x)-xd(g(x))-g(x)d(x)-xg(d(x)) for all x∈R. According to the relation [D(x),G(x)]=[d(x),g(x)] ,we obtain xd(g(x))-xg(d(x))=o for all x∈R. (20)*

Left –multiplying by y ,we get xd(g(x))y-xg(d(x))y =o for all x,y∈R. Since d

and g acts as a left centralizers , then $xd(g(x)y)-xg(d(x)y)=o$ for all $x,y\in R$.
$xd(g(x))y+xg(x)d(y)-xg(d(x))y-xd(x)g(y)=o$ for all $x,y\in R$.
According to (20) , we obtain
$x[g(x),d(x)]=o$ for all $x\in R$. (21)
Left –multiplying (21) by $d(r)$, we obtain
$d(r)x[g(x),d(x)]=o$ for all $x,r\in R$. Since d acts as a left centralizer and according to (21) , we obtain
$rd(x)[g(x),d(x)]=o$ for all $x,r\in R$. Replacing r by $g(x)$,we get
$g(x)d(x)\ [g(x),d(x)]=o$ for all $x\in R$. (22)
Left –multiplying (21) by $g(r)$,gives by same method
$d(x)\ g(x)[g(x),d(x)]=o$ for all $x\in R$. (23)
Subtracting (22) and (23) with using Lemmas (1 and 2) ,we obtain
$[g(x),d(x)]=o$ for all $x\in R$. By Theorem 2.1 ,we complets our proof.
By the same techniques in Corollary 2.2,we can prove the following corollary

Corollary2.8

Let R be a prime ring and U anon-zero ideal ,(D,d) and (G,g) be generalized derivations of R, if R admits to satisfy $[D(x),G(x)]=[d(x),g(x)]$ for all $x\in R$ and d and g acts as a left centralizers,then R is commutative.

Theorem 2.9

Let R be a prime ring,(D,d)and (G,g) be generalized derivations of R, if R admits to satisfy $[D(x),G(x)]=[d(x),g(x)]$ for all $x,y\in R$ and D acts as a left centralizer (resp. G acts as a left centralizer) . Then either d=0 or g=0.

Proof: We have $[D(x),G(x)]=[d(x),g(x)]$ for all $x\in R$.
Replacing x by xy ,we obtain $D(xy)G(xy)-G(xy)D(xy)=[d(xy),g(xy)]$ for all $x,y\in R$. Then
$D(xy)G(x)y+D(xy)xg(y)-G(x)yD(xy)-xg(y)D(xy)=[d(xy),g(xy)]$ for all $x,y\in R$.Since D acts as a left centralizer, then $D(x)yG(x)y+D(x)yxg(y)-G(x)yD(x)y-xg(y)D(x)y=[d(xy),g(xy)]$for all $x,y\in R$. Then
$[D(x)y,G(x)y]+D(x)yxg(y)-xg(y)D(x)y=[d(xy),g(xy)]$ for all $x,y\in R$.
$G(x)[D(x)y,y]+[D(x)y,G(x)]y+D(x)yxg(y)-xg(y)D(x)y=[d(xy),g(xy)]$ for all $x,y\in R$. Then
$G(x)D(x)y^2-G(x)yD(x)y+D(x)yG(x)y-G(x)D(x)y^2+D(x)yxg(y)-xg(y)D(x)y=[d(xy),g(xy)]$ for all $x,y\in R$. Then
$D(x)yG(x)y-G(x)yD(x)y+D(x)yxg(y)-xg(y)D(x)y=[d(xy),g(xy)]$ for all $x,y\in R$. (24)
Since D acts as a left centralizer ,we get
$D(xy)G(x)y-G(x)yD(xy)+D(xyx)g(y)-xg(y)D(xy)=[d(xy),g(xy)]$ for all $x,y\in R$. Then

D(x)yG(x)y-G(x)yD(x)y+D(xy)xg(y)-xg(y)D(x)y=[d(xy),g(xy)] for all x,y∈R.
D(x)yG(x)y-G(x)yD(x)y+D(x)yxg(y)+xd(y)xg(y)-xg(y)D(x)y=
[d(xy),g(xy)] for all x,y∈R.According to (24), above equation reduces to xd(y)xg(y)=o for all x,y∈R. Left –multiplying by zd(y),gives xd(y)xg(y)zd(y)=o for all x,y,z∈R. Rd(y)Rg(y)zd(y)=o . Since R is prime ring ,then either Rd(y)=0 or g(y)Rd(y)=o .Then in any case,we obtain either d=0 or g=0.
By the same techniques in Corollary 2.2,we can prove the following corollary

Corollary 2.10

Let R be a prime ring and U anon-zero ideal ,(D,d)and (G,g) be generalized derivations of R, if R admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x,y ∈R and D acts as a left centralizer (resp. G acts as a left centralizer) ,then R is commutative.

Theorem 2.11

Let R be a prime ring ,(D,d) and (G,g)be generalized derivations of R, if R admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x∈R, D and G acts as a left centralizers . Then either d=0 or g=0.

Proof: *We have [D(x),G(x)]=[d(x),g(x)] for all x∈R.*
Replacing x by xy , we get D(xy)G(xy)-G(xy)D(xy)=[d(xy),g(xy)]
for all x,y ∈R.
D(x) y G(xy) +xd(y)G(xy)-G(xy)D(x)y-G(xy)xd(y)=[d(xy),g(xy)] for all x,y∈R. (25)
Since D acts as a left centralizer ,we obtain
D(xy)G(xy)+xd(y)G(xy)-G(xy)D(xy)-G(xy)xd(y)=[d(xy),g(xy)] for all x,y∈R. Then
D(x)yG(xy)+xd(y)G(xy)+xd(y)G(xy)-G(xy)D(x)y-G(xy)xd(y)-
G(xy)xd(y)=[d(xy),g(xy)] for all x,y ∈R. According to (25) ,we obtain
xd(y)G(xy)-G(xy)xd(y)=o for all x,y∈R.
Since G acts as a left centralizer , we obtain
xd(y)G(xy)-G(xyx)d(y)=o for all x,y∈R. (26)
Then
xd(y)G(x)y+xd(y)xg(y)-G(xy)xd(y)-xyg(x)d(y)=o for all x,y∈R.
Since G acts as a left centralizer ,we get
xd(y)G(xy)-G(xyx)d(y)+xd(y)xg(y)-xyg(x)d(y)=o for all x,y∈R. According to (26), we obtain
xd(y)xg(y)-xyg(x)d(y)=o for all x,y ∈R. (27)
Left – multiplying (27) by D (r), we get
D(r)xd(y)xg(y)-D(r)xyg(x)d(y)=o for all x,y,r∈R. (28)
Since D acts as a left centralizer , we obtain

$D(rx)d(y)xg(y)-D(rx)yg(x)d(y)=o$ for all $x,y,r\in R$. Then $D(r)xd(y)xg(y)+rd(x)d(y)xg(y)-D(r)xyg(x)d(y)-rd(x)yg(x)d(y)=o$ for all $x,y,r\in R$. According to (28) , we get
$rd(x)[d(x),xg(x)]=o$ for all $x,r \in R$. Then $Rd(x)[d(x),xg(x)]=o$. Since R is semiprime ring ,then

$$d(x)[d(x),xg(x)]=o \text{ for all } x \in R . \qquad (29)$$

Left –multiplying (29) by $xg(x)$, we obtain

$$xg(x)d(x)[d(x),xg(x)]=o \text{ for all } x\in R. \qquad (30)$$

Left – multiplying (29)by $[d(x),xg(x)]$ and right – multiplying by $d(x)$ with using Lemmas (1 and 2), we obtain

$$[d(x),xg(x)]d(x)=o \text{ for all } x\in R. \qquad (31)$$

Left – multiplying (31)by $d(x)xg(x)$and right – multiplying by $xg(x)[d(x),xg(x)]$ with using Lemmas (1 and 2),we obtain

$$d(x)xg(x)[d(x),xg(x)]=o \text{ for all } x \in R. \qquad (32)$$

Subtracting (30) and (32) with using Lemmas (1 and 2), we get
$[d(x),xg(x)]=o$ for all $x\in R$. Then
$x[d(x),g(x)]+[d(x),x]g(x)=o$ for all $x\in R$. According to the relation $[D(x),G(x)]=[d(x),g(x)]$, we obtain

$$x[D(x),G(x)]+[d(x),x]g(x)=o \text{ for all } x\in R. \qquad (33)$$

Right- multiplying (33) by r, we get
$xD(x)G(x)r-xG(x)D(x)r+[d(x),x]g(x)r=o$ for all $x,r\in R$.
Since D acts as a left centralizer, we obtain
$xD(xG(x)r)-xG(x)D(xr)+[d(x),x]g(x)r=o$ for all $x,r\in R$.
Then $xD(x)G(x)r+x^2d(G(x)r)-xG(x)D(x)r-xG(x)xd(r)+[d(x),x]$
$g(x)r=o$ for all $x,r \in R$. According to (33)this equation become
$x^2d(G(x)r)-xG(x)xd(r)=o$ for all $x,r\in R$. Then

$$x^2d(G(x)r)+x^2G(x)d(r)-xG(x)xd(r)=o \text{ for all } x,r\in R. \qquad (34)$$

Since G acts as a left centralizer , we obtain
$x^2 d(G(x))r+x^2G(xd(r))-xG(x^2d(r))=o$ for all $x,r\in R$. Then
$x^2d(G(x))r+x^2G(x)d(r)+x^3g(d(r))-xG(x^2)d(r)-x^3g(d(r))=o$ for all $x,r\in R$. Then
$x^2d(G(x))r+x^2G(x)d(r)-xG(x)xd(r)-x^2g(x)d(r)=o$ for all $x,r\in R$. According to (34), this equation become $x^2g(x)d(r)=o$ for all $x,r\in R$. Left –multiplying by $xg(x)d(r)$ and right- multiplying by x with using Lemmas (1 and 2),we obtain
$xg(x)d(r)x=o$ for all $x,r\in R$. Right –multiplying by $g(x)d(r)$ with using Lemmas (1 and 2), we get
$xg(x)d(r)=o$ for all $x,r\in R$. Left –multiplying by $D(y)$,we obtain

$$D(y)xg(x)d(r)=o \text{ for all } x,r\in R. \qquad (35)$$

Since D acts as a left centralizer ,we obtain

$D(y)xg\ (x)d(r)+yd(x)g(x)d(r)=o$ for all $x,y,r\in R$.
According to (35),the equation become
$yd(x)g(x)d(r)=o$ for all $x,y,r\in R$. Replacing y by $g(x)$ and r by x with using Lemmas (1 and 2), we obtain
$g(x)d(x)=o$ for all $x\in R$. It is easy to get $g(x)Rd(x)=o$ .Then by primeness of R,we obtain either $d=0$ or $g=0$.
By the same techniques in Corollary 2.2,we can prove the following corollary

Corollary2.12

Let R be a prime ring and U anon-zero ideal ,(D,d) and (G,g)be generalized derivations of R, if R admits to satisfy $[D(x),G(x)]=[d(x),g(x)]$ for all $x\in R$, D and G acts as a left centralizers ,then R is commutative.

Theorem 2.13

Let R be a prime ring,(D,d)and (G,g) be generalized derivations of R, if R admits to satisfy $[D(x),G(x)]=[d(x),g(x)]$ for all $x\in R$. D and g acts as a left centralizers (resp. G and d acts as a left centralizers).Then either $d=0$ or $g=0$.

Proof: We have $[D(x),G(x)]=[d(x),g(x)]$ for all $x\in R$.
Replacing x by xy and since D and g acts as a left centralizers we obtain
$[D(x)y,G(xy)]=[d(xy)],g(xy)]$ for all $x,y\in R$. Then
$D(x)[y,G(xy)]+[D(x),G(xy)]y=g(x)[d(xy),y]+[d(xy),g(x)]y$ for all $x,y\in R$.
$D(x)[y,G(x)y]+D(x)[y,xg(y)]+[D(x),G(x)y]y+[D(x),xg(y)]y=g(x)[d(x)y,y]+g(x)[xd(y),y]+[d(x)y,g(x)]y+[xd(y),g(x)]y$ for all $x,y\in R$. Then
$D(x)[y,G(x)]y+D(x)x[y,g(y)]+D(x)[y,x]g(y)+G(x)[D(x),y]y+[D(x),G(x)]y^2+x[D(x),g(y)]y+[D(x),x]g(y)y=g(x)[d(x),y]y+g(x)x[d(y),y]+g(x)[x,y]d(y)+d(x)[y,g(x)]y+[d(x),g(x)]y^2+x[d(y),g(x)]y+[x,g\ (x)]d(y)y$ for all $x,y\in R$.
Replacing y by x and according to $[D(x),G(x)]=[d(x),g(x)]$, we obtain
$D(x)[x,G(x)]x+D(x)x[x,g(x)]+G(x)[D(x),x]x+x[D(x),g(x)]x+[D(x),x]g(x)x=g(x)[d(x),x]x+g(x)x[d(x),x]+d(x)[x,g(x)]x+x[d(x),g(x)]x+[x,g(x)]d(x)x$ for all $x\in R$. Then
$D(x)xG(x)x-D(x)G(x)x^2+D(x)x^2g(x)-D(x)xg(x)x+G(x)D(x)(x^2)-G(x)xD(x)x+xD(x)g(x)x-xg(x)D(x)x+D(x)xg(x)x-xD(x)g(x)x=g(x)d(x)x^2-g(x)xd(x)x+g(x)xd(x)x-g(x)x^2d(x)+d(x)xg(x)x-d(x)g(x)x^2+xd(x)g(x)x-xg(x)d(x)x+xg(x)d(x)x-g(x)xd(x)x$ for all $x\in R$.
According to $[D(x),G(x)]=[d(x),g(x)]$,we obtain
$D(x)xG(x)x+D(x)x^2g(x)-G(x)xD(x)x-xg(x)D(x)x=-g(x)x^2d(x)+d(x)xg(x)x+xd(x)g(x)x-g(x)xd(x)x$ for all $x\in R$. Then
$D(x)xG(x^2)-G(x^2)D(x)x=d(x)xg(x)x-g(x)x^2d(x)+xd(x)g(x)x-g(x)xd(x)x$ for all $x\in R$. (36)

Since D and g acts as left centralizers , we obtain

$D(x^2)G(x^2)-G(x^2)D(x^2)=d(x)xg(x^2)-g(x^3)d(x)+xd(x)g(x^2)-g(x^2)d(x)x$ *for all* $x\in R$.

Since D acts as left centralizer , then

$D(x)xG(x^2)-G(x^2)D(x)x=d(x)xg(x^2)-g(x^3)d(x)+xd(x)g(x^2)-g(x^2)d(x)x$ *for all* $x\in R$. (37)

Since g acts as left centralizer , (37) become

$D(x)xG(x^2)-G(x^2)D(x)x=d(x)xg(x)x-g(x^2)xd(x)+xd(x)g(x)x-g(x)xd(x)x$ *for all* $x\in R$. (38)

Substituting (36)in (38), we obtain

$-g(x)x^2d(x)=-g(x^2)xd(x)$ *for all* $x\in R$. *Then*

$xg(x)xd(x)=o$ *for all* $x\in R$.

Since g acts as left centralizer , then

$x^2g(x)d(x)=o$ *for all* $x\in R$. (39)

Left – multiplying (39) by $xg(x)d(x)$*and right –multiplying by x with using Lemmas (1 and 2) , we obtain*

$xg(x)d(x)x=o$ *for all* $x\in R$. (40)

Right –multiplying (40)by $g(x)d(x)$ *with using Lemmas (1 and 2),we get*

$xg(x)d(x)=o$ *for all* $x\in R$. (41)

Left –multiplying (41)by D(r), we obtain

$D(r)xg(x)d(x)=o$ *for all* $x,r\in R$.

Since D acts as a left centralizer , we get $D(rx)g(x)d(x)=o$ *for all* $x,r\in R$. *Then* $rd(x)g(x)d(x)=o$ *for all* $x,r\in R$. *Replacing r by g(x) with using Lemmas (1 and 2), we obtain*

$g(x)d(x)=o$ *for all* $x\in R$. *It is easy we get* $d(x)Rg(x)=o$. *Then by primeness of R,we obtain either d=0 or g=0.*

We closed our paper by the following corollary,which can be prove by the same techniques in Corollary 2.2.

Corollary2.14

Let R be a prime ring and U a non-zero ideal ,(D,d)and (G,g) be generalized derivations of R, if R admits to satisfy $[D(x),G(x)]=[d(x),g(x)]$ *for all* $x\in R$. *D and g acts as a left centralizers (resp. G and d acts as a left centralizers),then R is commutative.*

حول الاشتقاقات العامة مع الابدالية على الحلقات الاولية

محسن جبل عطية
الجامعة المستنصرية-كلية التربية- قسم الرياضيات

الملخص :ان الغرض الرئيسي من هذا البحث هو دراسة وتحري بعض النتائج بخصوص الاشتقاقات العامة مع الابدالية على الحلقات الاولية

عندما تكون الدالة التجمعية تعمل بفاعلية وكـأنها مركزي ايسر على R R

سوف نحصل على الدالة التجميعية (احداهما $g=0$ او $d=0$ d)

=صفر والحلقة الاولية عندماتحوي مثالي غير صفري

تكون ابدالية .

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