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Study of Some Dynamical Concepts in General Topological spaces

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Abstract. In this paper some famous dynamical concepts such as sensitive, transitive , mixing and equicontinuous are studied in a topological $\mathbb{G}-$space and the essential properties of these maps are proved. Furthermore, the concept of Devaney's chaotic maps in topological $\mathbb{G}-$spaces is studied.

Key Words. topological $\mathbb{G}-$spaces, sensitive , transitive , mixing, equicontinuous, expansive, chaotic maps.

Mathematics subject classification : 37B05

1.Introduction. The chaotic dynamical systems has been studied in detail in the past [1, 2, 3, 4, 5, 6] . Any chaotic system is established on the notion of irregularity and can not be disintegrated into two invariant open subsets . There are numerous definitions of chaos famous but Devaney's definition of chaos [4] for discrete dynamical systems is most widespread and commonly customary definition. In [5] Tian and Chen have defined and considered Devaney's chaos for a sequence of functions in iterative and successive ways on a metric space and have given numerous curious examples to illustrate such chaotic systems . In [7], uniform convergence, mixing and chaos are considered . In [6], Devenay's chaos of uniform limit functions is considered . In [8, 9, 10] topological transitivity of uniform limit functions is studied . In [11], the notation of topological transitivity of uniform limit functions on metric $G-$spaces is considered . In [12] different types of convergence of sequence of real valued functions have been considered.

A topological $\mathbb{G}-$ space is a triple $(X,\mathbb{G},\theta)$, where X is a topological space (not necessarily metrizable space) , $\mathbb{G}$ is a topological group and $\theta:\mathbb{G}\times X\to X$ is a continuous action of $\mathbb{G}$ on X . For $S\subset X, g\in\mathbb{G}$,

$$gS=\{\theta(g,s)\ |s\in S\}.$$

For $\in X$, the set

$$\mathbb{G}(x)=\{\theta(g,x)\ |g\in\mathbb{G}\},$$

is called the $G-$*orbit* of x in X. For every $x,y \in X$ either $\mathbb{G}(x)\cap\mathbb{G}(y)=\emptyset$ or $\mathbb{G}(x)=\mathbb{G}(y)$. A subset S of X is called $G-$invariant if $\theta(\mathbb{G}\times S)\subseteq S$. Let

$$X/\mathbb{G}=\{\mathbb{G}(x): x\in X\}$$

and $p_X: X\to X/\mathbb{G}$ be the natural quotient map taking x to $\mathbb{G}(x), x\in X$ then $X/\mathbb{G}$ donated with the quotient topology is called the orbit space of X (with respect to $\mathbb{G}$) . The map p_X is also a closed map . By trivial action of $\mathbb{G}$ on X we mean $\theta(g,x)$ for all $\in\mathbb{G}$, $x\in X$. If X,Y are $\mathbb{G}-$spaces, then a continuous map $h: X\to Y$ is called *equivariant* if $h(\theta(g,x))=\theta(g,h(x))$ for each $g\in\mathbb{G}$ and each $x\in X$. The continuous map h *is called pseudo-equivariant* if $h(\mathbb{G}(x))=\mathbb{G}(h(x))$ for each $x\in X$. An equivariant map is pseudo-equivariant but converse is not true [7] . In [7, 9, 10] numerous curious consequences using pseudo-equivariant maps have been obtained . The orbit [4] of f at a point $x\in X$ is defined as follows

$$O(f,x)=\{f^n(x): n\geq 0\}.$$

Let f be a function from a $\mathbb{G}-$space X into itself. We define the $\mathbb{G}_f-$orbit [11] of f at a point $x\in X$ as follows

$$\mathbb{G}-O(f,x)=\{\theta(g,f^n(x)): g\in G, n\geq 0\}.$$

A topological space X is said to be **Urysohn space** [21] if for every $x,y\in X$ with $x\neq y$ there exist couple of open sets M and N such that $x\subset M, y\subset N$ and $\overline{M}\cap\overline{N}=\phi$.

R. Das in [17] present the concept of the chaos of a sequence of maps in a metric $G-$space .

R. Das in [18] gave sufficient conditions under which product of two maps, in which one is Devaney's $\mathbb{G}_1-$chaotic and other is Devaney's G_2-chaotic, is Devaney's $\mathbb{G}_1\times\mathbb{G}_2-$ chaotic .

V. Kumar, in [19] introduce chaos in topological spaces and study some properties of chaos spaces which include hyper spaces .

S.H.Abd and I.J.Kadhim in [20] they define expansive maps in general topological space (not necessarily metrizable space) and generalize this definition to $\mathbb{G}-$spaces and give some properties of such maps . Also they study some properties of new type of chaotic maps which called $\mathbb{G}-$expansive chaotic maps.

The goal of this paper is to define sensiti-ve, transitive, mixing and equicontinuous maps in topological $\mathbb{G}-$space and study the Devaney's chaotic in topological $\mathbb{G}-$space .

2. $\mathbb{G}-$Periodic Points

Here the definition of periodic points in $\mathbb{G}-$space is stated and some essential properties are proved.

Definition 2.1 [11] A point x in a topological space X is said to be periodic point of a map $f: X\to X$ if $f^n(x)=x$ for some $n\in\mathbb{N}$. The smallest positive integer n satisfy $f^n(x)=x$ is called period of f .

Definition 2.2 [11 A point $x\in X$ is said to be $\mathbb{G}-$periodic point of a map f from a $\mathbb{G}-$ space into itself if $f^n(x)=\theta(g,x)$ for some $n\in\mathbb{N}$ and $g\in\mathbb{G}$.The periodic of f is the smallest positive integer n satisfy $f^n(x)=\theta(g,x)$ ▌

Remark 2.3 The concepts of periodic point and $\mathbb{G}-$periodic point are coincide Under trivial action of $\mathbb{G}$ on X . Under non-trivial action of $\mathbb{G}$ on X, if $x\in X$ is periodic point of $f: X\to X$, then it is $\mathbb{G}-$periodic point. Nonetheless the converse need not be true as we see in the next example .

Example 2.4[20] Let $X=[0,1]$ under the usual topology, $\mathbb{G}=\{-1,1\}$ under the discrete topology. Define $\theta:\mathbb{G}\times X\to X$ by $\theta(-1,x)=1-x$ and $\theta(1,x)=x$, $x\in X$. Then θ is a continuous action of $\mathbb{G}$ on X. Let $T: X\to X$ be the tent map. Then $x=\frac{1}{7}$ is $\mathbb{G}-$peridic point but not periodic point .

For the proof of the following three statement see [18,20].

Proposition 2.5 Let $(X,\mathbb{G}_1,\theta_1)$ be $\mathbb{G}_1-$space and $(Y,\mathbb{G}_2,\theta_2)$ be $\mathbb{G}_2-$ space. Let $f: X\to X$ and $h: Y\to Y$ be equivariant functions. Then $x\in X$ is $\mathbb{G}_1-$periodic point of f and $x\in Y$ is $\mathbb{G}_2-$periodic point of h if and only if $(x,y)\in X\times Y$ is $\mathbb{G}_1\times\mathbb{G}_2-$periodic point of $f\times h$.

Corollary 2.6 Let $(X, \mathbb{G}_1, \theta_1)$ be $\mathbb{G}_1$ −space and $(Y, \mathbb{G}_2, \theta_2)$ be $\mathbb{G}_2$ − space. Let $f: X \to X$ and $h: Y \to Y$ be equivariant maps. If $x \in X$ is $\mathbb{G}_1$ −periodic point of f of period $n_1$and $x \in Y$ is G_2 −periodic point of h of period n_2 then $(x, y) \in X \times Y$ is $\mathbb{G}_1 \times \mathbb{G}_2$ −periodic point of $f \times h$ with period $n = lcm(n_1, n_2)$.

Theorem 2.7 Let $(X, \mathbb{G}_1, \theta_1)$ be $\mathbb{G}_1$ −space and $(Y, \mathbb{G}_2, \theta_2)$ be $\mathbb{G}_2$ − space. Let $f: X \to X$ and $h: Y \to Y$ be equivariant maps. If the set $\mathbb{G}_1$ −periodic point of f is dense in X and the set of $\mathbb{G}_2$ −periodic point of h is dense in Y then the set of $\mathbb{G}_1 \times \mathbb{G}_2$ −periodic point of $f \times h$ is dense in $X \times Y$.

3. G − Sensitive Maps

V. Kumar in [19] define the sensitivity of maps in topological spaces. In this section we shall define the sensitivity of maps in $\mathbb{G}$− spaces and give some properties of such maps .

Definition 3.1[19] Let X be a topological space. A continuous function $f: X \to X$ is said to be f **sensitive** at $x \in X$, if given any open set U with $x \in U$ then there exist $y \in U$, $n \in \mathbb{N}$ and an open V such that $f^n(x) \in V$ and $f^n(y) \notin \overline{V}$. We say that f is sensitive if it is sensitive at every points of X .

Now we shall introduce the following definition .

Definition 3.2 Let X be a $\mathbb{G}$ −space. We say that $f: X \to X$ is a continuous map is $\mathbb{G}$− **sensitive** at $x \in X$, if given any open set U with $x \in U$ then there exist $y \in U$ with $\mathbb{G}(x) \neq \mathbb{G}(y)$, $n \in \mathbb{N}$ and an open V such that $f^n(u) \in V$ and $f^n(v) \notin \overline{V}$ for all $u \in \mathbb{G}(x), v \in \mathbb{G}(y)$.

Remark 3.3 The concepts of sensitive and $\mathbb{G}$ − sensitive are agreed under the trivial action of $\mathbb{G}$ on X.

Examples 3.4

(a) In indiscrete space and discrete space there are no ($\mathbb{G}$ −)sensitive maps.

(b) Constant maps are not ($\mathbb{G}$ −)sensitive.

(c) Let $X = [0,1]$ with the usual topology, consider the tent function $T: X \to X$ define by $T(x) = 1 - 2|x - 1|$ and $\mathbb{G} = \{-1,1\}$ be discrete group with the action $\theta: G \times X \to X$ define by $\theta(-1, x) = 1 - x, \theta(1, x) = x$ for all $x \in X$. We shall show that the tent map is $\mathbb{G}$ −sensitive at $x = 0$. The collection of all open sets containing $x = 0$ is given by $\{[0, a), 0 < a < 1\} \cup \{[0,1]\}$.If $M = [0,1]$, then $0 \in U$. Let $y = \frac{1}{3}$, then $y \in M$. Set $N = [0, \frac{1}{2})$, then N is open set in X. Since $\mathbb{G}(x) = \{\theta(g, x): g \in G\}$, then $\mathbb{G}(0) = \{0,1\}$, and $\mathbb{G}(\frac{1}{3}) = \{\frac{1}{3}, \frac{2}{3}\}$. It follows that $\mathbb{G}(0) \neq \mathbb{G}(\frac{1}{3})$. Since $T(\theta(g, 0)) = 0 \in N$ for every $g \in \mathbb{G}$ and $T\left(\theta(g, \frac{1}{3})\right) = \frac{2}{3} \notin \overline{N}$ for every $g \in \mathbb{G}$ .If $M = [0, a)$ with $0 < a < \frac{1}{2}$. Set $N = [0, a)$ and let $0 < y < a$. Then there exists $n \in \mathbb{N}$ such that $T^n(\theta(g, 0)) = 0 \in N$ for every $g \in \mathbb{G}$ and $T^n(\theta(g, y)) \notin \overline{N}$ for every $g \in \mathbb{G}$. If $M = [0, a)$ with $\frac{1}{2} \leq a < 1$. Set $N = [0, \frac{1}{2})$ and let $y = \frac{1}{2}$. Then $T(\theta(g, 0)) = 0 \in N$ for every $g \in \mathbb{G}$ and $T(\theta(g, y)) = 1 \notin \overline{N}$ for every $g \in \mathbb{G}$. Thus we get the result. In a similar way we can show that the tent map is $\mathbb{G}$ −sensitive at every $x \in X$.

Theorem 3.5 Let X, Y be two $\mathbb{G}$ −spaces and $f_1: X \to X$, $f_2: Y \to Y$ be equivariant topologically conjugate via $\varphi: X \to Y$. If f_1 is $\mathbb{G}$ −sensitive at $x \in X$, then f_2 is $\mathbb{G}$ −sensitive at $\varphi(x)$.

Proof Suppose that f_1 is $\mathbb{G}$ − sensitive at $x \in X$. Let V be an open set in Y containing $y = \varphi(x)$.Since φ is topological conjugacy then $U_1 = \varphi^{-1}(U_2)$ is an open set in X containing x. By hypothesis there exist $x' \in U_1$with $\mathbb{G}(x) \neq \mathbb{G}(x')$, $n \in \mathbb{N}$ and an open set V_1 such that $f_1^n(u) \in V_1$ and $f_1^n(u') \notin \overline{V_1}$ for all $u \in \mathbb{G}(x), u' \in \mathbb{G}(x')$. Since φ is equivariant, then it is pseudo equivariant. Therefore $\varphi(\mathbb{G}(x)) = \mathbb{G}(\varphi(x)) = \mathbb{G}(y)$. Also $\left(\mathbb{G}(x')\right) = \mathbb{G}(\varphi(x')) = \mathbb{G}(y')$. Since $\mathbb{G}(x) \neq \mathbb{G}(x')$ and φ is bijective, then $\mathbb{G}(y) \neq \mathbb{G}(y')$.Set $V_2 = \varphi(V_1)$, then V_2 is an open set in X.Let $v \in \mathbb{G}(y)$ and $v' \in \mathbb{G}(y')$, then $\varphi^{-1}(v) \in \mathbb{G}(x)$ and $\varphi^{-1}(v') \in G(x')$.Thus

$f_2^n(\varphi(\varphi^{-1}(v_1)) \in V_2$ and $f_2^n(\varphi(\varphi^{-1}(v_2)) \notin \overline{V_2}$ .i.e., $f_2^n(v_1) \in V_2$ and $f_2^n(v_2) \notin \overline{V_2}$.This mean that h_2 is $\mathbb{G}-$sensitive at $\varphi(x)$. ■

Corollary 3.6 Let X, Y be two topological spaces and $f_1: X \to X$, $f_2: Y \to Y$ be topologically conjugate via $\varphi: X \to Y$. If f_1 is sensitive at $x \in X$, then f_2 is sensitive at $\varphi(x)$.

Theorem 3.7 Let X be a $\mathbb{G}_1-$space , Y be $\mathbb{G}_2-$space and $f_1: X \to X$, $f_2: Y \to Y$. If either f_1 is G_1-sensitive or f_2 is $\mathbb{G}_2-$sensitive, then $f_1 \times f_2$ is $\mathbb{G}_1 \times \mathbb{G}_2-$sensitive .

Proof Let $z = (x_1, y_1) \in X \times Y$ and let U be an open set containing z. Then $x_1 \in X, y_1 \in Y$ and there exist open set U_1 in X containing $x_1$and U_2 in Y containing y_1 such that $U_1 \times U_2 \subseteq U$. Since f_1 is $\mathbb{G}_1-$sensitive then there exists $x_2 \in U_1$ with $\mathbb{G}(x_1) \neq \mathbb{G}(x_2)$, $n \in \mathbb{N}$ and an open set V_1 in X such that $f_1^n(u_1) \in V_1$ and $f_1^n(u_2) \notin \overline{V_1}$ for all $u_1 \in \mathbb{G}(x_1), u_2 \in \mathbb{G}(x_2)$. Now for any $y_2 \in U_2$, $w = (x_2, y_2) \in U_1 \times U_2 \subseteq U$, we have $\mathbb{G}_1 \times \mathbb{G}_2(x_{1,}y_1) \neq \mathbb{G}_1 \times \mathbb{G}_2(x_{2,}y_2)$ and for all $g = (g_1, g_2), q = (q_1, q_2) \in G = \mathbb{G}_1 \times \mathbb{G}_2$ we have

$$(f_1 \times f_2)^n(\theta(g, z)) = f_1^n \times f_2^n(\theta(g, z))$$

$$= \left(f_1^n(\theta_1(g_1, x_1)), f_2^n(\theta_2(g_2, y_1)\right)$$

$$\in V_1 \times Y = V$$

and

$$(f_1 \times f_2)^n(\theta(q, w)) = f_1^n \times f_2^n(\theta(q, w))$$

$$= \left(f_1^n(\theta_1(q_1, x_2)), f_2^n(\theta_2(q_2, y_2)\right)$$

$$\notin \overline{V_1} \times Y = \overline{V}. \blacksquare$$

Corollary 3.8 Let X and Y be two topological spaces and $f_1: \to X$, $f_2: Y \to Y$. If either f_1 is sensitive or f_2 is sensitive, then $f_1 \times f_2$ is sensitive .

Definition 3.9 [19] A non-empty subset F of a topological space X is said to be attracting to the continuous map $f: F \to F$ if $f^n(x) \in F$ for all non-negative integer n, $x \in F$.

Definition 3.10 A non-empty subset F of a $\mathbb{G}-$space X is said to be **attracting** to the continuous map $f: F \to F$ with respect to the action $\theta: \mathbb{G} \times X \to X$ if $f^n(\theta(g, x)) \in F$ for all non-negative integer n , $x \in F$ and $g \in \mathbb{G}$.

Remark 3.11 Under trivial action of G on X the notions of attracting and $\mathbb{G}-$ attracting are coincided .

Example 3.12 Let $X = \mathbb{R}$ $\mathbb{G} = \{-1,1\}$ and $\theta: \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ be an action of $\mathbb{G}$ on $\mathbb{R}$ defined by $\theta(-1, x) = 1 - x, \theta(1, x) = x$. Then $(\mathbb{R}, \mathbb{G}, \theta)$ is $G-$space. If $F = [0,1]$, then F is attracting to the tent map with respect to the given action . While, $W = [0, \frac{1}{2}]$ is not attracting to the tent map with respect to the same action .

Theorem 3.13 Let X be a Urysohn $\mathbb{G}-$space and F be a perfect attracting subset of X of a continuous map $f: F \to F$ with the property that f^n not constant for some $n \geq 1$, then f is $\mathbb{G}-$sensitive on F.

Proof Let $x \in F$ and U be an open set in X containing x . Then $U \cap F \neq \phi$. Hence x is an adherence point. Since F is perfect, then F "cannot have an isolated point, thus x is not isolated point and consequently x is a limit point of F. Then $(U - \{x\}) \cap F \neq \phi$. Hence there exists $y \in (U - \{x\}) \cap F$. Thus $x \neq y$, then $\mathbb{G}(x) \neq \mathbb{G}(y)$(since θ is effective). Since F is attracting to f with respect to θ, then $f^n(\theta(g, x)) \in F$ for all non-negative integer n. Now, $f^n(\theta(g, x)) \neq f^n(\theta(q, y))$, for some n and all $g, q \in \mathbb{G}$ Otherwise f^n becomes constant on $U \cap F$.Since X is Urysohn, there exists an open set V such that $f^n(\theta(g, x)) \in V$ and $f^n(\theta(q, y)) \notin \overline{V}$ for all $g, q \in \mathbb{G}$. Thus all f is all $\mathbb{G}-$ sensitive on all F.■

Corollary 3.14. Let X be a Urysohn $\mathbb{G}-$space and $f: X \to X$ be a continuous map with the property that f not constant. Then $f \in \mathbb{G} - S(F)$iff F is perfect .

Example 3.15. Consider Example 2.4(c). According to the Example 1.11 and Theorem 2.12, the tent map is $\mathbb{G}-$sensitive on all $F = [0,1]$.■

Definition 3.16 [20] If X is a topological space and $h \in \mathcal{H}(X)$ then h is called **expansive**, if for every $x, y \in X$ with $x \neq y$, then there exists an open set V in X and a positive integer n such that $f^n(x) \in V$ and $f^n(y) \notin \overline{V}$.

Definition 3.17[20] If X is a $\mathbb{G}$ −space and $h \in \mathcal{H}(X)$ then h is called $\mathbb{G}$ −**expansive**, if there exists an open set V in X such that whenever $x, y \in X, \mathbb{G}(x) \neq \mathbb{G}(y)$ then there exists an integer n satisfying $f^n(u) \in V$ and $f^n(v) \notin \overline{V}$, for all $u \in \mathbb{G}(x)$ and $v \in \mathbb{G}(y)$.

Remark 3.18 Under the trivial action of $\mathbb{G}$ on X the notions of expansive and $\mathbb{G}$ − expansive are coincided. It is observed that the notion of expansiveness and the notation of $\mathbb{G}$ −expansive under a nontrivial action of $\mathbb{G}$ are independent of each other .

Proposition 3.19 Every ($\mathbb{G}$ −)expansive map is ($\mathbb{G}$ −)sensitive .

Proof Clear.

4. $\mathbb{G}$ − equicontinuos Maps

In this section the notion of equicontinuous mapping is defined in general topological space and some essential results are proved.

Definition 4.1 [19] A mapping $f: X \to X$ is said to be equicontinuos at a point $x \in X$ if for every neighborhood V of $f(x)$ and for every positive integer n, there exists a neighborhood U of x such that $f^n(U) \subseteq V$.

Clearly that any self-continuous map is equicontinuos but the converse need not be true .

Definition 4.2 Let X be a $\mathbb{G}$ − space. A mapping $f: X \to X$ is said to be $\mathbb{G}$ − equicontinuos at a point $x \in X$ if for every neighborhood V of $f(\theta(g, x))$ and for every positive integer n, there exists a neighborhood U of x such that $f^n(\theta(g, U)) \subseteq V$ for all $g \in \mathbb{G}$.

Remark 4.3 The concepts of equicontinuous and $\mathbb{G}$ − equicontinuous are agreed under the trivial action of $\mathbb{G}$ on X .

Theorem 4.4 Let X , Y be $\mathbb{G}$ − spaces and $h_1: X \to X$, $h_1: Y \to Y$ be equivariant topologically conjugate via $\varphi: X \to Y$. If h_1 is $\mathbb{G}$ −equicontinuous, then so is h_2.

Proof. Suppose that h_1 is $\mathbb{G}$ −equicon-tinuous. Let $y \in Y$ and V_2 be a neighborhood of $h_2(\theta(g, y))$. Since φ is onto, then there exists $x \in X$ such that $\varphi(x) = y$. Since φ is equivariant topologically conjugacy, then so is φ^{-1} . Thus

$$\varphi^{-1}\big(h_2(\theta(g, y))\big) = h_1(\theta(g, \varphi^{-1}(y)))$$

$$= h_1(\theta(g, x)).$$

Therefore $h_1(\theta(g, x)) \in V_1$ where $V_1 = \varphi^{-1}(V_2)$ .Hence φ is homeomorphism and consequently V_1 is a neighborhood of $h_1(\theta(g, x))$. But h_1 is $\mathbb{G}$ −equicontinuous, there exists a neighborhood U_1 of x such that $h_1^n(\theta(g, U_1)) \subseteq V_1$ for all $g \in \mathbb{G}$ and all integer n. Then $\varphi(h_1^n(\theta(g, U_1))) \subseteq \varphi(V_1)$ for all $g \in G$ and all integer n. Since φ is equivariant topological conjugacy, we have $h_2^n(\theta(g, U_2)) \subseteq V_2$ for all $g \in \mathbb{G}$ and all integer n,where $U_2 = \varphi(U_1)$ is a neighborhood of y. Then h_2 is $\mathbb{G}$ −equicontinuous. ■

Corollary 4.5 Let $h_1: X \to X$ and $h_1: Y \to Y$ be topologically conjugate via $\varphi: X \to Y$. If h_1 is equicontinuous, then so is h_2.

Theorem 4.6 Let , Y be $\mathbb{G}$ − spaces and $h_1: X \to X$, $h_1: Y \to Y$ be maps. Then $h_1 \times h_2: X \times Y \to X \times Y$ is $\mathbb{G}_1 \times \mathbb{G}_2$ − equicontinuous iff h_1 is $\mathbb{G}_1$ − equicontinuous and h_2 is $\mathbb{G}_2$ −equaicontinuous.

Proof. Suppose that $h_1 \times h_2$ is $\mathbb{G}_1 \times \mathbb{G}_2$ −equicontinuous. We shall show that h_1 is $\mathbb{G}_1$ −equicontionuos and similarly we can show that h_2 is $\mathbb{G}_2$ −equaicontinuous. Let $x \in X$, V be neighborhood of $h_1(\theta_1(g, x))$ and n be any positive integer. If $y \in Y$ then $(x, y) \in X \times Y$. Since Y is a neighborhood of $h_2(\theta_2(g, y))$ for every $g \in \mathbb{G}$, then $V \times Y = W$ is a neighborhood of

$h_1 \times h_2(\theta((g,q),(x,y)))$.

By hypothesis, there exists a neighborhood U of (x,y) such that $(h_1 \times h_2)^n(\theta(\bar{g},U)) \subseteq V$ for all $\bar{g} \in \mathbb{G}_1 \times \mathbb{G}_2$.Since U be a neighborhood of (x,y), then there exists two neighborhoods U_1 and U_2 of x and y respectively such that $U_1 \times U_2 \subseteq U$. Thus we have

$h_1^n(\theta(g,U_1)) \times h_2^n(\theta(q,U_2)) \subseteq V \times Y$ for all $g \in \mathbb{G}_1, q \in \mathbb{G}_2$. Hence

$h_1^n(\theta(g,U_1)) \subseteq V$ for all $g \in \mathbb{G}_1$.

This means that h_1 is $\mathbb{G}_1$ −equicontinuous .

Conversely, suppose that h_1 is $\mathbb{G}_1$ − equico-ntinuous and h_2 is $\mathbb{G}_2$ −equaicontinuous. Let $(x,y) \in X \times Y$ and be a nhd of

$(h_1 \times h_2)\ (\theta((g,q),(x,y)))$.

Then there exists two nhd's V_1 and V_2 of $h_1(\theta_1(g,x))$ and $h_2(\theta_2(q,y))$ respectively such that $V_1 \times V_2 \subseteq V$. By hypothesis, there exist two neighborhoods U_1 and U_2 of x and y respectively such that

$h_1^n(\theta_1(g,U_1)) \subseteq V_1$ for all $g \in \mathbb{G}_1$ and $h_2^n(\theta_2(q,U_2)) \subseteq V_2$ for all $q \in \mathbb{G}_2$. Set $U_1 \times U_2 = U$. Thus we have

$(h_1 \times h_2)^n(\theta(\bar{g},U)) \subseteq V$

for all $\bar{g} \in \mathbb{G}_1 \times \mathbb{G}_2$.This means that $h_1 \times h_2 : X \times Y \to X \times Y$ is $\mathbb{G}_1 \times \mathbb{G}_2$ −equicontinu-ous. ■

Corollary 4.7 Suppose that X, Y be spaces and $h_1 : X \to X$, $h_1 : Y \to Y$ be maps. Then $h_1 \times h_2 : X \times Y \to X \times Y$ is equicontinuous iff both h_1 and h_2 are equicontinuous.

Theorem 4.8 Let X be a Urysohn $\mathbb{G}$ −space with a finite group G and $f \in \mathcal{H}(X)$. If f is G −equicontinuous map, then its G −expan-sive.

Proof. Suppose that f is $\mathbb{G}$ −equsicontinuous. Let $x,y \in X$ with $\mathbb{G}(x) \neq \mathbb{G}(y)$. Since f is equivariant and one-to-one, then $\mathbb{G}(f(x)) \neq \mathbb{G}(f(y))$. Since X is Urysohn and $\mathbb{G}(f(x))$, $\mathbb{G}(f(y))$ are disjoint sets in X then there exist two open sets V_g and W_g such that $\theta(g,f(x)) \in V_g, \theta(g,f(y)) \subset W_g$ and $\overline{V_g} \cap \overline{W_g} = \phi$.Set $\Omega = \{V_g : g \in \mathbb{G}\}$ and $\Psi = \{W_g : g \in \mathbb{G}\}$. Then Ω, Ψ are finite collections of open sets. Set $V = \cup_{g\in G} V_g$ and $W = \cup_{g\in G} W_g$ then V and W open sets containing $\mathbb{G}(f(x))$, $\mathbb{G}(f(y))$ respectively. Since Ω, Ψ are finite collections then $\overline{V} = \cup_{g\in G} \overline{V_g}$, $\overline{W} = \cup_{g\in G} \overline{W_g}$ and $\overline{V} \cap \overline{W} = \phi$.By hypothesis f is $\mathbb{G}$ −equicontinuous on X, then its $\mathbb{G}$ −equicontinuous at every point in X.Therefore there exist tow open sets U_1 and U_2 containing x and y respectively such that

$f^n(\theta(g,U_1)) \subset V$ and $f^n(\theta(g,U_2)) \subset W$

for every $g \in \mathbb{G}$ and every integer n. We have

$f^n(\theta(g,x)) \in V$ and $f^n(\theta(q,y)) \notin \overline{V}$ for every integer n. Let $u \in \mathbb{G}(x)$ and $v \in \mathbb{G}(y)$. Then there exist $g,q \in \mathbb{G}$ such that $u = \theta(g,x), v = \theta(q,y)$. Thus we have $f^n(u) \in V$ and $f^n(v) \notin \overline{V}$ for every $u \in \mathbb{G}(x)$ and $v \in \mathbb{G}(y)$. This means that f is $\mathbb{G}$ −expansive. ■

Corollary 4.9 Let X be a Urysohn $\mathbb{G}$ −space with a finite group $\mathbb{G}$ and $f \in \mathcal{H}(X)$. If f is $\mathbb{G}$ −equicontinuous map, then its $\mathbb{G}$ −sensitive .

Theorem 4.10 Let X be a Urysohn space and $f \in \mathcal{H}(X)$. If f is equicontinuous map, then its expansive .

Proof. Suppose that f is equicontinuous. Let $x,y \in X$ with $x \neq y$. Since f is equivariant and one-to-one, then $f(x) \neq f(y)$. Since X is Urysohn then there exist tow open sets V and W such that $f(x) \in V, f(y) \subset W$ and $\overline{V} \cap \overline{W} = \phi$.By hypothesis f is equicontinuous on X, then its equicontinuous at every point in X.Therefore there exist tow open sets U_1 and U_2 containing x and y respectively such that

$$f^n(U_1) \subset V \text{ and } f^n(U_2) \subset W$$

for every $g \in G$ and every integer n.Thus we have

$$f^n(x) \in V \text{ and } f^n(y) \notin \overline{V} \ \forall n \in \mathbb{Z}.$$

This means that f is expansive. ■

Here we shall generalize Theorem 4.11.

Theorem 4.11 Let X be a compact Hausdorff $\mathbb{G}-$space with a compact group G and $f \in \mathcal{H}(X)$. If f is $\mathbb{G}-$equicontinuous map, then its $\mathbb{G}-$expansive .

Proof Suppose that f is $\mathbb{G}-$equsicontinuous. Let $x, y \in X$ with $\mathbb{G}(x) \neq \mathbb{G}(y)$. Since f is equivariant and one-to-one, then $\mathbb{G}(f(x)) \neq \mathbb{G}(f(y))$. Since X is T_2 and G is compact then $G(x), G(y)$ are closed sets in X. But $f \in \mathcal{H}(X)$ so $\mathbb{G}(f(x)), \mathbb{G}(f(y))$ are disjoint closed sets in X. Since X is compact and T_2 then it is normal, thus there exist two disjoint open sets V and W containing $\mathbb{G}(f(x)), \mathbb{G}(f(y))$ respectively such that $\overline{V} \cap \overline{W} = \phi$.By hypothesis f is $\mathbb{G}-$equicontinuous on X, then its $\mathbb{G}-$equicontinuous at every point in X.Therefore there exist tow open sets U_1 and U_2 containing x and y respectively such that

$$f^n(\theta(g, U_1)) \subset V \text{ and } f^n(\theta(g, U_2)) \subset W$$

for every $g \in \mathbb{G}$ and every integer n. For every integer n we have $f^n(\theta(g, x)) \in V$ and $f^n(\theta(q, y)) \notin \overline{V}$.Let $u \in \mathbb{G}(x)$ and $v \in \mathbb{G}(y)$. Then there exist $g, q \in G$ such that $u = \theta(g, x), v = \theta(q, y)$. Hence $f^n(u) \in V$ and $f^n(v) \notin \overline{V}$ for every $u \in \mathbb{G}(x)$ and $v \in \mathbb{G}(y)$ and consequently f is $\mathbb{G}-$expansive. ■

Corollary 4.12 Let X be a compact Hausdorff $\mathbb{G}-$space with a compact group G and $f \in \mathcal{H}(X)$. If f is $\mathbb{G}-$equicontinuous map, then its $\mathbb{G}-$sensitive .

Theorem 4.13 Let X be a Urysohn $\mathbb{G}-$space .If f is equivariant $\mathbb{G}-$equicontinuous map without fixed point then it is not $\mathbb{G}-$transitive .

Proof Let $x \in X$. Since f has no fixed point then $f(\theta(g, x)) \neq x$ for every $g \in \mathbb{G}$. Set $y = f(\theta(g, x))$ then $x \neq y$. Since X be Urysohn space then there exist pair of disjoint open sets U and V such that $x \subset U, y \subset V$ and $\overline{U} \cap \overline{V} = \phi$. Now since $f(\theta(g, x)) \in V$ and f is $\mathbb{G}-$equicontinu-ous on X,then it is $\mathbb{G}-$ equicontinuous at $\theta(g, x)$for every $g \in \mathbb{G}$ therefore there exists open neighborhood N of x such that $f^n(\theta(g, N)) \subset V$ for every integer n. Hence $f^n(\theta(g, N)) \cap U = \phi$ for all n. But f is equivalent, then $f^n(\theta(g, N)) = \theta(g, f^n(N))$ for all n.Hence $\theta(g, f^n(N)) \cap U = \phi$ for all n .This means that f is not $\mathbb{G}-$transitive. ■

Corollary 4.14 Let X be a Urysohn space .If f is equivariant equicontinuous map without fixed point then it is not transitive .

Theorem 4.15 Let X be a $\mathbb{G}-$space. If an onto mapping $f: X \to X$ is $\mathbb{G}-$equicontinuous, then it is not $\mathbb{G}-$sensitive .

Proof Suppose that f is $\mathbb{G}-$equicontinuous. Let $y \in X$ and let V be an open set containing y. Since f is onto, there exists $x \in X$ such that $y = f(x)$. By hypothesis there exists neighborhood U of x such that $f^n(\theta(g, U)) \subset V$ for every $g \in \mathbb{G}$ and every integer n. Let $z \in U$, then $f^n(\theta(g, z)) \in f^n(\theta(g, U)) \subset V \subset \overline{V}$ for every $g \in \mathbb{G}$. Thus for every open set V in X, there exists $x \in X$ and an open set U such that for every $z \in U$ and an integer n we have $f^n(\theta(g, x)) \in V$ and $f^n(\theta(g, z)) \in \overline{V}$.This means that f is not $\mathbb{G}-$ sensitive.■

5. Chaotic Maps on $\mathbb{G}-$spaces

In this final section the chaotic maps on $\mathbb{G}-$space is studied where the phase space X is any topological space(not necessarily metrizable). First we state the definitions of $\mathbb{G}-$transitive map and $\mathbb{G}-$mixing [18].

Definition 5.1.[18] Let X be a topological space . A continuous map $f: X \to X$ is said to be topologically transitive (or transitive) if for every pair of non-empty open subsets U and V of X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \phi$

Definition 5.2 [18] Let X be a $\mathbb{G}$ −space . A continuous map $f: X \to X$ is said to be $\mathbb{G}$ −transitive if for every pair of non-empty open subsets U and V of X, there exists $n \in \mathbb{N}$ and $g \in \mathbb{G}$ such that $\theta(g, f^n(U)) \cap V \neq \phi$.

Remark 5.3 Under trivial action of $\mathbb{G}$ on X, notions of transitive and $\mathbb{G}$ −transitive coincide. Under non-trivial action of $\mathbb{G}$ on X, if f is transitive then it is$\mathbb{G}$ −transitive. But the following example shows that every $\mathbb{G}$ −transitive map need not be transitive .

Example 5.4 [18] Let $X = \left\{\mp\frac{1}{n}, \mp\left(1-\frac{1}{n}\right): n \in \mathbb{N}\right\}$ under usual topology. Consider action of $\mathbb{Z}_2$, additive group of integers mod 2, on X given by $\theta(0,t) = t$ and $\theta(1,t) = 1 - t$, $t \in X$. Then $f: X \to X$ defined by

$$f(x) = \begin{cases} x_+, & if\ x \in \left\{\frac{1}{n}, 1-\frac{1}{n}: n \neq 1, n \in \mathbb{N}\right\} \\ x_- & if\ x \in \left\{-\frac{1}{n}, -\left(1-\frac{1}{n}\right): n \neq 1, n \in \mathbb{N}\right\} \\ x & if\ x \in \{-1,0,1\} \end{cases}$$

where x_+ (resp. x_-) denotes element of X immediate to right (resp. to left) of x. then f is $\mathbb{Z}_2$ −transitive but not transitive.

Definition 5.5 [18] Let X be a $\mathbb{G}$ −space. A continuous map $f: X \to X$ is said to be $\mathbb{G}$ −mixing if for every pair of non-empty open subsets U and V of X, there exists $n_0 \in \mathbb{N}$ and $g \in \mathbb{G}$ such that for all $n \geq n_0$, $\theta(g, f^n(U)) \cap V \neq \phi$.

Remark 5.6[18] Every topologically $\mathbb{G}$ −mixing map is topologically $\mathbb{G}$ −transitive.

Theorem 5.7 [18] Let X be a $\mathbb{G}_1$ −space , Y be $\mathbb{G}_2$ −space and $f_1: X \to X$, $f_2: Y \to Y$ be continuous maps. If f_1 is topologically $\mathbb{G}_1$ −mixing and f_2 is topologically $\mathbb{G}_2$ −mixing, then $f_1 \times f_2$ is topologically $\mathbb{G}_1 \times \mathbb{G}_2$ − mixing.

Now, we shall introduce the notion of $\mathbb{G}$ − chaotic map in general topological space.

Definition 5.8. Let X be a $\mathbb{G}$ −space and F be a compact subset of X. A continuous map $f: X \to X$ is said to be $\mathbb{G}$ −chaotic on F if

(i) f is $\mathbb{G}$ −transitive, **(ii)** The $\mathbb{G}$ −periodic points of f are dense in F,**(iii)** f is $\mathbb{G}$ −sensitive on F.

Notation 5.9

(i) $\mathbb{G}C(F) = \{f: F \to F: f \text{ is } \mathbb{G} - \text{chaotic on } F\}$.

(ii) $\mathbb{G}CH(X) = \{F \in K(X): \mathbb{G}C(F) \neq \phi\}$.

Definition 5.10 A $\mathbb{G}$ −space X is said to be $\mathbb{G}$ −chaos space if $GCH(X) \neq \phi$. In this case the elements of $\mathbb{G}CH(X)$ are called $\mathbb{G}$ −chaotic sets.

Remark 5.11 By Example 3.4 , if a $\mathbb{G}$ −space X is discrete or indiscrete space, then it is not $\mathbb{G}$ −chaos space .

In the following examples we shall show that the statements (i),(ii) and (iii) in Definition 6.1 are independent and not two of them imply the other .

Example 5.12 In this example we shall show that (i) and (ii) $\not\Rightarrow$(iii). Let $X = \{0,1\}$, $\tau = \{\phi, \{0\}, X\}$ and $\mathbb{G} = \{-1,1\}$ under the discrete topology with the action $\theta: \mathbb{G} \times X \to X$ defined by $\theta(-1,x) = 1 - x$, $\theta(1,x) = x$, $x \in X$. Define $f: X \to X$ by $f(0) = 0$, $f(1) = 1$ (the identity map). Then f is continuous on X. It is easy to see that

(a) f is $\mathbb{G}$ −transitive; (b) $\overline{GPr(f,X)} = X$;

(c) f is not $\mathbb{G}$ −sensitive on X.

Example 5.13 In this example we shall show that (ii) and (iii) $\not\Rightarrow$(i). Let $X = [0,1]$ under the usual topology and $\mathbb{G} = \{-1,1\}$ under the discrete topology with the action $\theta: \mathbb{G} \times X \to X$ defined by $\theta(-1,x) = 1 - x$, $\theta(1,x) = x$, $x \in X$. Define $f: X \to X$ by $f(x) = x$ (the identity map). Then f is continuous on X. It is easy to see that

(a) f is not $\mathbb{G}$ −transitive; (b) $\overline{\mathbb{G}Pr(f,X)} = X$;

(c) f is $\mathbb{G}$ −sensitive on X.

Example 5.14 In this example we shall show that (i) and (iii) $\nRightarrow$(ii) Let $X = [0, 3/4]$ under the usual topology and $\mathbb{G} = \{-1,1\}$ under the discrete topology with the action $\theta: \mathbb{G} \times X \to X$ defined by $\theta(-1, x) = 1 - x$, $\theta(1, x) = x$, $x \in X$. Define $f: X \to X$ by

$$f(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \le x \le 1/2 \\ \frac{3}{2}(1-x), & \text{if } 1/2 \le x \le 3/4 \end{cases}.$$

Then f is homeomorphism on X. It is easy to see that

(a) $\overline{\mathbb{G}O(f,x)} = X$, for all $x \in X$; (b) $\overline{\mathbb{G}Pr(f,X)} \ne X$.

(c) f is $\mathbb{G} -$ sensitive on X.

Example 5.15 In this example we need to show that (iii) $\nRightarrow$(ii) and (i). For, let $X = \mathbb{R}$ and $F = [-1,1]$ under the usual topology and $\mathbb{G} = \mathbb{Z}_2$ under the discrete topology with the action $\theta: \mathbb{G} \times X \to X$ defined by $\theta(0, x) = x$, $\theta(1, x) = 1 - x$, $x \in X$. Define $f: X \to X$ by $(x) = x^2$. Then f is continuous on X. It is easy to see that

(a) f is not $\mathbb{G}$ −transitive on F. (b) Since the $(f, F) = \{0,1\}$, then $\overline{\mathbb{G}Pr(f,F)} \ne F$. (c) f is $\mathbb{G}$ −sensitive on F.

Theorem 5.16 A $\mathbb{G}$ −chaos space is topological property.

Proof We want to prove that if X is $\mathbb{G}$ −chaos space and if X and Y are homeomorphic, then Y is $\mathbb{G}$ −space. Let $\varphi: X \to Y$ be a homeomorphism. Since X is a $\mathbb{G}$ −space, there exists $F \in \mathbb{G}CH(X)$. So $\mathbb{G}C(F) \ne \phi$. Let $f \in \mathbb{G}C(F)$. Let $h = \varphi \circ f \circ \varphi^{-1}$. Then h is continuous function from $\varphi(F)$ onto $\varphi(F)$. We prove that $h \in \mathbb{G}C(\varphi(F))$ so that $\varphi(F) \in \mathbb{G}CH(Y)$.i.e., we have to prove that

(i) h is $\mathbb{G}$ −transitive, **(ii)** the set of all $\mathbb{G}$ −periodic points of h are dense in F and **(iii)** h is $\mathbb{G}$ −sensitive on $\varphi(F)$.

Proof of (i) Let U and V be two non-empty sets in $\varphi(F)$. Then $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ be two non-empty sets in F. Since f is $\mathbb{G}$ −transitive on F, then there exists $n \in \mathbb{N}$, $g \in \mathbb{G}$ such that

$$\theta_1(g, f^n(\varphi^{-1}(U))) \cap \varphi^{-1}(V) \ne \phi.$$

So $\theta_1(g, \varphi^{-1}(h^n(U))) \cap \varphi^{-1}(V) \ne \phi$ which implies $\varphi^{-1}(\theta_2(g, (h^n(U))) \cap \varphi^{-1}(V) \ne \phi$, hence $\theta_2(g, (h^n(U))) \cap V \ne \phi$. This means that h is $\mathbb{G}$ −transitive.

Proof of (ii). Since $F \in \mathbb{G}CH(X)$, then $\overline{\mathbb{G}Pr(f,F)} = F$. Suppose, if possible, that $\overline{\mathbb{G}Pr(h,\varphi(F))} \ne \varphi(F)$. Then there is an open set V in $\varphi(F)$ such that $\mathbb{G}Pr(h, \varphi(F)) \cap V = \phi$. Let $y \in \mathbb{G}Pr(h, \varphi(F))$, then $y \in \varphi(F)$ such that $h^n(y) = \theta(g, y)$ for some $n \in \mathbb{N}$ and $g \in \mathbb{G}$. Since y is $\mathbb{G}$ −periodic points of h, then $x = \varphi^{-1}(y)$ is G −periodic point of f. Now, we have $y = \varphi(x) \notin V$ which implies

$\theta(g, y) \notin \theta(g, V)$, then $h^n(y) \notin \theta(g, V)$ and so $\varphi \circ f^n \circ \varphi^{-1}(\varphi(x)) \notin \theta(g, V)$. Therefore

$\varphi(f^n(x)) \notin \theta(g, V)$ which implies

$f^n(x) \notin \varphi^{-1}(\theta(g, V)) = \theta(g, \varphi^{-1}(V))$, i.e.

$\theta(g, x) \notin \theta(g, \varphi^{-1}(V))$, so $x \notin \varphi^{-1}(V)$,

which is a contradiction .

Proof of (iii). Let $y_1 \in \varphi(F)$. Then $y_1 = \varphi(x_1)$ for some $x_1 \in F$. Let V be an open nhd of y_1. Then $\varphi^{-1}(V)$ is open in F and $\varphi^{-1}(V)$ is a neighborhood of x_1. Since $f \in GS(F)$, there exists $x_2 \in \varphi^{-1}(V) \cap F$ with $G(x_1) \ne G(x_2)$, $n \in \mathbb{N}$ and an open set U such that

$f^n(\theta(g, x_1)) \in U$ and $f^n(\theta(q, x_2)) \notin \overline{U}$.

Since $f^n(\theta(g, x_1)) \in U$, then

$\varphi(f^n(\theta(g, x_1))) \in \varphi(U)$

so $\varphi \circ f^n \circ \varphi^{-1} \circ \varphi(\theta(g, x_1))) \in \varphi(U)$

hence

$\varphi \circ f^n \circ \varphi^{-1}(\theta(g, \varphi(x_1)))) \in \varphi(U)$,i.e. $h^n(\theta(g, y_1)) \in \varphi(U)$.

Similarly, since $f^n(\theta(q,x_2)) \notin \overline{U}$, then

$$\varphi(f^n(\theta(q,x_2))) \notin \varphi(\overline{U}) = \overline{\varphi(U)}$$

so

$$\varphi \circ f^n \circ \varphi^{-1} \circ \varphi(\theta(q,x_2))) \notin \overline{\varphi(U)}$$

hence

$$\varphi \circ f^n \circ \varphi^{-1}(\theta(q,\varphi(x_2)))) \notin \overline{\varphi(U)},$$

i.e. $h^n(\theta(g,y_2)) \in \overline{\varphi(U)}$, where $y_2 = \varphi(x_2)$.

Since φ is homeomorphism, θ is equvariant and $\mathbb{G}(x_1) \neq \mathbb{G}(x_2)$, then $\mathbb{G}(y_1) \neq \mathbb{G}(y_2)$.■

Theorem 5.17 Let X be a $\mathbb{G}_1$ −space, Y be a $\mathbb{G}_2$ − space and $f: X \to X$, $h: Y \to Y$ be equivariant maps. If f is Devaney's $\mathbb{G}_1$ −chaotic, topologically $\mathbb{G}_{2_1}$ −mixing and h is Devaney's $\mathbb{G}_2$ −chaotic, topologically $\mathbb{G}_2$ −mixing then $f \times h$ is Devaney's $\mathbb{G}_1 \times \mathbb{G}_2$ −chaotic.

Proof By Theorem 3.7 , $f \times h$ is $\mathbb{G}_1 \times \mathbb{G}_2$ − sensitive. By Theorem 2.7 , $f \times h$ has dense $\mathbb{G}_1 \times \mathbb{G}_2$ − periodic points. By Theorem 5.3 , the map $f \times h$ is $\mathbb{G}_1 \times \mathbb{G}_2$ −topologically mixing and hence $\mathbb{G}_1 \times \mathbb{G}_2$ − topologically transitive. Thus $f \times h$ is Devaney's $\mathbb{G}_1 \times \mathbb{G}_2$ − chaotic .

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دراسة بعض المفاهيم الديناميكية في الفضاءات التبولوجية العامة

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سمه كاظم جبر
جامعة القادسية
كلية الزراعة
قسم علوم التربة والموارد المائية

المستخلص :

في هذا البحث تم دراسة بعض المفاهيم الديناميكية في الفضاءات التبولوجية العامة مثل ، الحساسية و التعدي و الخلط و التطبيقات المتساوية الاستمر ارية. بالاضافة الى دراسة مفهوم ديفيني للفوضى في الفضاءات التبولوجية العامة .