Derivation of composite rules for numerical calculation of double integrals using Romberg acceleration to improve results

Presented by

Rehab Ali Kudear (Lecture)

University of Kufa, College of Education for Girls, Department of Mathematics

Abstract

The main goal of this work is to find numerical value of a double integral where the integrand is continuous, by derivation composite rules and its correction terms. Depending on these terms we improve these values by using Romberg acceleration, after that we used these rules to find the values of integral with continuous integrands on the region of integral and we compare these results with composite rule of trapezoidal on the two dimensions (exterior and interior), the resulting values are better if accuracy and number of sub-intervals are taken in account.

1-Introduction:

Finding the value of double integral is much more complicated than that of a single one, this is because the former depends on two variables as well as the requirement of continuity and singularity of the integrand, beside the singularity of the partial derivatives of the integrand which lead to great difficulties, bearing in mind that we deal with the regions or surfaces and not with intervals as is the case in single integration. Except some rear cases, it is not easy to solve numerically these types of double integrals. The importance of these integrals appears in evaluation of surfaces, the central means and moments of inertia for plane surfaces, in addition to that, to find the volume under the double integral plane. As an example, the volume yields from rotation of heart curve $\rho = 2(1-\cos\theta)$ round the polar coordinates, or to evaluate the volume of the sphere $x^2 + y^2 + z^2 = 36$ inside of the cylinder $y^2 + z^2 = 6y$. Ayers [1]

These led many researchers to work in the field of numerical solution of double integrals. Among those Hass and Jacobson [2], who through lights on solving integrals with continues integrands of the expression $f(x,y) = f_1(x)\, f_2(y)$

Here, we derive combined rules (base on suggested [3] and trapezoidal methods) to find the correction terms for the double integrals. We applied these rules on double integrals in which the integrand are continues and bounded on integration intervals. These simple methods gave high accuracy and relatively fast results. We applied the Romberg acceleration convergence method which gave advantage in comparison with trapezoidal method on the dimensions: interior (x) and exterior (y). This process will be symbolized by TT(Mohammad [4])

2- Singular integral for continuous integrand [5,6]

Suppose that J is defined as follows :

$$J = \int_{x_0}^{x_n} f(x)\,dx = G(h) + E_G(h) + R_G \qquad \ldots(1)$$

Such that $f(x)$ is a continuous integrand lies above the x-axis on the interval $[x_0, x_n]$, $G(h)$ represents Largranian approximation of the value of integration, *E(h)* is a series of correction terms that can be added to *G (h)* , *J* represents the area under the curve $y = f(x)$ and above x-axis and bounded by the parallel lines $x = x_n$, $x = x_0$, the general form of $G(h)$ is given by

$$G(h) = h(w_0 f_0 + w_1 f_1 + w_2 f_2 + ... + w_1 f_{n-1} + w_0 f_n \ ) \qquad ...(2)$$

where w_i are weighted factors , and $f_r = f(x_r)$, $h = \frac{x_n - x_0}{n}$ $x_r = x_0 + rh$, $r = 0,1,2,..., n$

to simplify formula (2) we rewrite the weights in terms of w_0 provided that $w_1 = 2(1 - w_0), w_2 = 2w_0$. Now, if we let $w_0 = 1/2$, then we will get the trapezoidal rule and then symbolized to the function $G(h)$ by the symbol

$$T(h) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + ... + f_n)$$

And when $w_0 = 0$ we get the mid-point rule and symbolized by the symbol $M(h)$:

$M(h) = h(f_1 + f_3 + f_5 + ... + f_{n-1})$, where *n* is the number sub-intervals.

The general formula for the suggested method (which depend on the rules of trapezoid and the mid-point [3]) symbolized by *Su* is

$$Su = \frac{h}{4}\left(f_0 + f_n + 2f(x_0 + (n - \frac{1}{2})h) + 2\sum_{i=1}^{n-1}(f_i + f(x_0 + (i - \frac{1}{2})h) \right)$$

Such that $f_i = f(x_0 + ih)$, $i = 1,2,..., n$. To find the correction terms $E_G(h)$ see reference [5,6]

The reminder $R_G(h)$ has the form $R_G = \frac{2^n}{(2k)} B_{2k} \ h^{2k+1} \ f^{(2k)}(\lambda)$,where $x_0 \prec \lambda \prec x_n$ is Bernoulli number

3- Romberg integration [7,8]

The Romberg method is an application of Ralston method to find the best value for J using the trapezoid, mid-point and suggested rules.

Suppose that we applied the error formulas for two different values of h , say h_1, h_2, we find that

$$J - G(h_1) = A_G h_1^2 + B_G h_1^4 + C_G h_1^6 + \ ... \qquad ...(3)$$

$$J - G(h_2) = A_G h_2^2 + B_G h_2^4 + C_G h_2^6 + ... \qquad ...(4)$$

where $A_G , B_G ,...$ are constants .

Substituting $h_2 = \frac{1}{2}h_1$ in the formula (4), and solving it together with formula (3) for A_G and neglecting those terms which contain $h^4, h^6,...$ from both mentioned formulas we will get

$$J \cong \left(\frac{2^2 \, G(h/2) - G(h)}{(2^2 - 1)} \right). \qquad ...(5)$$

Where $h = h_1$.

Formula (5) does not represent the accurate value of integration, but it is to some extend closer to the real value of the integration than the two values $G(h/2), G(h)$, it will be symbolized by

$$G(h, h/2) = \left(\frac{2^2 \, G(h/2) - G(h)}{(2^2 - 1)} \right) \qquad ....(6)$$

Thus,

$$J - G(h, h/2) = A'_G \ h_1^4 + B'_G h_1^6 + ... \qquad ...(7)$$

where $A'_G , B'_G ,...$ are constants.

In a similar way a closer value of the integration can be found using $G(h, h/2)$, and hence we get a table of values of Romberg table and in general the values of this table can be calculated using

$$G = \frac{2^k\, G(h/2) - G(h)}{2^k - 1} \qquad ...(8)$$

where $k = 2,4,6,...$, and G is the value of a new column of Romberg table, and $G(h/2), G(h)$ are present in the previous column, the first column of Romberg table represents the use of the suggested method on the inner dimension x and the external dimension y, which symbolized by *SuSu*, and applying the suggested method based on the inner dimension x and the rule of trapezoidal on the external dimension y, which symbolized by *TSu*, and the use of trapezoidal rule on the internal dimension of x and the suggested method on the external dimension y, which symbolized by *SuT*, and finely the value of Romberg table is determined according to the required accuracy which we call Eps, in which the relative error is as follows

$\left|\frac{G_2 - G_1}{G_1}\right| \le Eps \quad , G_1 \ne 0$, where $G_2 \;, G_1$ are two approximate values of the integral in a single row of Romberg table with a method of numerical integration.

4.<u>Derivation of composite rules to calculate continuous double integrals and formulas for the error using the trapezoidal rule and the suggested method</u>

Suppose that the integral I is defined as follows

$$I = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

which can be written as

$$I = \int_c^d \int_a^b f(x,y)\, dx\, dy = GG(h) + E_{GG}(h) + R_{GG}(h) \qquad ...(9)$$

where $GG(h)$ is an approximate value representing the integral using one of the methods $SuT, TSu, SuSu$, and that $E_{GG}(h)$ is series of possible correction terms added to the values $GG(h)$ will divide the integral interval on the internal dimension [a, b] for a number of sub-intervals (n) , and divide the integral interval on the external dimension [c, d] for a number of sub-intervals (m), where $\bar{h} = (d-c)/m$, $h = (b-a)/n$. We let $h = \bar{h}$ to be able to use Romberg acceleration.

First: the suggested rule on the two dimensions (internal x and external y) ($SuSu$)

It is known that

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

can be written on the internal dimension (x) with the suggested method as follows

$$Su = \int_a^b f(x,y)\, dx = \frac{h}{4}\left(f(a,y) + f(b,y) + 2f(a+(n-0.5)h, y) + 2\sum_{i=1}^{n-1} \left(f(x_i,y) + f(x_0 + (i-0.5)h, y)\right)\right)$$
$$+ \frac{b-a}{24} h^2 \frac{\partial^2 f(\lambda_1, y)}{\partial x^2} - \frac{b-a}{1440} h^4 \frac{\partial^4 f(\lambda_2, y)}{\partial x^4} + \frac{61(b-a)}{60480} h^6 \frac{\partial^6 f(\lambda_3, y)}{\partial x^6} + \dots \qquad ...(10)$$

where $\lambda_k \in (a,b)$, $k = 1, 2, 3, \dots$, $x_i = a + ih, i = 1,2,3,...,n-1$ and $h = (b-a)/n$.

Integrating both sides of (10) numerically for y on the interval [c, d] using the suggested method yields

$$SuSu=\int_c^d\int_a^b f(x,y)\,dxdy=\frac{h^2}{16}\Big[\,f(a,c)+f(a,d)+f(b,c)+f(b,d)+2\big(\,f(a,c+(n-0.5)h$$
$$+f(b,c+(n-0.5)h)\big)+f(a+(n-0.5)h,c)+f(a+(n-0.5)h,d)+f(a+(n-0.5)h,c+(n-0.5)h)$$
$$+2\sum_{i=1}^{n-1}\,(f(a,y_i)+f(a,c+(i-0.5)h)+f(b,y_i)+f(b,c+(i-0.5)h+2f(a+(n-0.5)h,y_i)+$$
$$2f(a+(n-0.5)h,c+(i-0.5)h)+f(x_i,c)+f(x_i,d)+2f(x_i,c+(n-0.5)h)+f(a+(i-0.5)h,c)+$$
$$f(a+(i-0.5)h,d)+2f(a+(i-0.5)h,c+(n-0.5)h))+4\sum_{j=1}^{n-1}\,(f(x_i,y_j)+f(x_i,c+(j-0.5)h)$$
$$+f(a+(i-0.5)h,y_j)+f(a+(i-0.5)h,c+(j-0.5)h))\,\Big]+(b-a)(d-c)$$
$$(\frac{1}{24}h^2\frac{\partial^2 f(\lambda_1,\theta_1)}{\partial x^2}-\frac{1}{1440}h^4\frac{\partial^4 f(\lambda_2,\theta_2)}{\partial x^4}+\frac{61}{60480}h^6\frac{\partial^6 f(\lambda_3,\theta_3)}{\partial x^6}+\ldots)+h^2(\frac{h}{4}\frac{d-c}{24}\frac{\partial^2 f(a,\mu_{11})}{\partial y^2}+$$
$$\frac{h}{4}\frac{d-c}{24}\frac{\partial^2 f(b,\mu_{21})}{\partial y^2}+\ldots)+h^4(-\frac{h}{4}\frac{d-c}{1440}\frac{\partial^4 f(a,\mu_{12})}{\partial y^4}-\frac{h}{4}\frac{d-c}{1440}\frac{\partial^4 f(b,\mu_{22})}{\partial y^4}+\ldots)$$

where $\theta_i\in(c,d)$, $i=1,2,3,\ldots$; and $\mu_{mn}\in(c,d)$, $m=1,2,3,\ldots,n+1$, $n=1,2,3,\ldots$

Since $\frac{\partial^4 f}{\partial y^4},\frac{\partial^6 f}{\partial y^6},\ldots$ and $\frac{\partial^4 f}{\partial x^4},\frac{\partial^6 f}{\partial x^6},\ldots$ are continuous functions at each point of the region $[a,b]\times[c,d]$,it means that the value of integral (9) using the rule $SuSu$ with the correction terms are:

$$SuSu=\int_c^d\int_a^b f(x,y)\,dxdy=\frac{h^2}{16}\Big[\,f(a,c)+f(a,d)+f(b,c)+f(b,d)+2(f(a,c+(n-0.5)h$$
$$+f(b,c+(n-0.5)h))+f(a+(n-0.5)h,c)+f(a+(n-0.5)h,d)+f(a+(n-0.5)h,c+(n-0.5)h)$$
$$+2\sum_{i=1}^{n-1}\,(f(a,y_i)+f(a,c+(i-0.5)h)+f(b,y_i)+f(b,c+(i-0.5)h+2f(a+(n-0.5)h,y_i)+$$
$$2f(a+(n-0.5)h,c+(i-0.5)h)+f(x_i,c)+f(x_i,d)+2f(x_i,c+(n-0.5)h)+f(a+(i-0.5)h,c)+$$
$$f(a+(i-0.5)h,d)+2f(a+(i-0.5)h,c+(n-0.5)h))+4\sum_{j=1}^{n-1}\,(f(x_i,y_j)+f(x_i,c+(j-0.5)h)$$
$$+f(a+(i-0.5)h,y_j)+f(a+(i-0.5)h,c+(j-0.5)h))\,\Big]+A_{SuSu}h^2+B_{SuSu}h^4+C_{SuSu}h^6+\ldots$$

where B_{SuSu},C_{SuSu} A_{SuSu} are constants whose values depend on the partial derivatives of the function $f(x,y)$,and that $i=1,2,3\ldots,n-1$, $x_i=a+ih$, $j=1,2,3,\ldots,n-1$, $y_j=c+jh$

<u>Second:-</u> the suggested method base on the external dimension y and trapezoidal rule on the internal dimension x (SuT **)**

Applying the trapezoidal method on single integral $\int_a^b f(x,y)\,dx$ gives

$$T=\int_a^b f(x,y)dx=\frac{h}{4}(f(a,y)+f(b,y)+2\sum_{i=1}^{n-1}f(x_i,y))-\frac{b-a}{12}h^2\frac{\partial^2 f(\lambda_1,y)}{\partial x^2}+\frac{b-a}{720}h^4\frac{\partial^4 f(\lambda_2,y)}{\partial x^4}$$
$$-\frac{b-a}{30240}h^6\frac{\partial^6 f(\lambda_3,y)}{\partial x^6}+\ldots \qquad \ldots.(11)$$

whereas $\lambda_k \in (a,b)$, $k=1,2,3,...$; $x_i = a+ih$, $h=(b-a)/n$, and integrating both sides of (11) numerically for y on the interval [c, d] using the suggested method, also we will get

$$SuT = \int_c^d \int_a^b f(x,y)\,dxdy = \frac{h^2}{8}\Big[\ f(a,c)+f(a,d)+f(b,c)+f(b,d)+2(f(a,c+(n-0.5)h+f(b,c+(n-0.5)h))$$
$$+2\sum_{i=1}^{n-1}(f(a,y_i)+f(a,c+(i-0.5)h)+f(b,y_i)+f(b,c+(i-0.5)h)+f(x_i,c)+f(x_i,d)$$
$$+2f(x_i,c+(n-0.5)h))+4\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(f(x_i,y_j)+f(x_i,c+(j-0.5)h))\ \Big]+(b-a)(d-c)\,(\frac{-1}{12}h^2\frac{\partial^2 f(\lambda_1,\theta_1)}{\partial x^2}+$$
$$\frac{1}{720}h^4\frac{\partial^4 f(\lambda_2,\theta_2)}{\partial x^4}-\frac{1}{30240}h^6\frac{\partial^6 f(\lambda_3,\theta_3)}{\partial x^6}+...)+h^2(\frac{h}{4}\frac{d-c}{24}\frac{\partial^2 f(a,\mu_{11})}{\partial y^2}+\frac{h}{4}\frac{d-c}{24}\frac{\partial^2 f(b,\mu_{21})}{\partial y^2}$$
$$+\frac{h}{2}\sum_{i=1}^{n-1}\frac{d-c}{24}\frac{\partial^2 f(x_i,\mu_{2+i\ 1})}{\partial y^2})+h^4(-\frac{h}{4}\frac{5\,(d-c)}{1440}\frac{\partial^4 f(a,\mu_{12})}{\partial y^4}-\frac{h}{4}\frac{5(d-c)}{1440}\frac{\partial^4 f(b,\mu_{22})}{\partial y^4}+...)$$

where $\theta_i \in (c,d)$, $i=1,2,3,...$.

Hence, we note that the value of integration (9) using SuT with the correction terms will take the following form

$$SuT = \int_c^d \int_a^b f(x,y)\,dxdy = \frac{h^2}{8}\Big[\ f(a,c)+f(a,d)+f(b,c)+f(b,d)+2(f(a,c+(n-0.5)h+f(b,c+(n-0.5)h))$$
$$+2\sum_{i=1}^{n-1}(f(a,y_i)+f(a,c+(i-0.5)h)+f(b,y_i)+f(b,c+(i-0.5)h)+f(x_i,c)+f(x_i,d)$$
$$+2f(x_i,c+(n-0.5)h))+4\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(f(x_i,y_j)+f(x_i,c+(j-0.5)h))\ \Big]++A_{SuT}h^2+B_{SuT}h^4+C_{SuT}h^6+...$$

where B_{SuSu}, C_{SuSu} A_{SuSu} are constants whose values depend on the partial derivatives of the function $f(x,y)$,and that $i=1,2,3...,n-1$, $x_i = a+ih$, $j=1,2,3,...,n-1$, $y_j = c+jh$

<u>Third:</u> - the suggested method rule on internal dimension x and trapezoidal rule on the external dimension (TSu) **:**

Applying the suggested method in single integration $\int_a^b f(x,y)dx$ we get on

$$Su = \int_a^b f(x,y)\,dx = \frac{h}{4}\left(f(a,y)+f(b,y)+2f(a+(n-0.5)h,y)+2\sum_{i=1}^{n-1}(f(x_i,y)+f(x_0+(i-0.5)h,y)\right)$$
$$+\frac{b-a}{24}h^2\frac{\partial^2 f(\lambda_1,y)}{\partial x^2}-\frac{b-a}{1440}h^4\frac{\partial^4 f(\lambda_2,y)}{\partial x^4}+\frac{61(b-a)}{60480}h^6\frac{\partial^6 f(\lambda_3,y)}{\partial x^6}+... \qquad ...(12)$$

where $\lambda_k \in (a,b)$, $k=1,2,3,...$,and that $x_i = a+ih$ and $h=(b-a)/n$ and integrating both sides of the equations (12) numerically for y in the interval [c, d] using the trapezoidal method, we get to:

$$TSu=\int_c^d\int_a^b f(x,y)\,dxdy=\frac{h^2}{8}[\ f(a,c)+f(a,d)+f(b,c)+f(b,d)+2(f(a+(n-0.5)h,c)$$

$$+f(a+(n-0.5)h,d))+2\sum_{i=1}^{n-1}(f(a,y_i)+f(b,y_i)+2f(a+(n-0.5)h,y_i)+f(x_i,c)$$

$$+f(x_i,d)++f(a+(i-0.5)h,c)+f(a+(i-0.5)h,d))+4\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(f(x_i,y_j)+f(a+(i-0.5)h,y_j))\ ]$$

$$+(b-a)(d-)(\frac{1}{24}h^2\frac{\partial^2 f(\lambda_1,\theta_1)}{\partial x^2}+-\frac{1}{1440}h^4\frac{\partial^4 f(\lambda_2,\theta_2)}{\partial x^4}+\frac{1}{60480}h^6\frac{\partial^6 f(\lambda_3,\theta_3)}{\partial x^6}+...)+$$

$$h^2(\frac{-h}{4}\frac{d-c}{12}\frac{\partial^2 f(a,\mu_{11})}{\partial y^2}-\frac{h}{4}\frac{d-c}{12}\frac{\partial^2 f(b,\mu_{21})}{\partial y^2}+\frac{h}{2}\frac{d-c}{12}\frac{\partial^2 f(a+(n-0.5)h,\mu_{31})}{\partial y^2}+$$

$$\frac{h}{2}\sum_{i=1}^{n-1}\frac{d-c}{-12}\frac{\partial^2 f(x_i,\lambda_{2+i\ 1})}{\partial y^2})+h^4(-\frac{h}{4}\frac{(d-c)}{720}\frac{\partial^4 f(a,\mu_{12})}{\partial y^4}-\frac{h}{4}\frac{(d-c)}{720}\frac{\partial^4 f(b,\mu_{22})}{\partial y^4}+...)$$

where $\theta_i\in(c,d)$, $i=1,2,3,...$ Hence, we note that the value of integration (9) using TSu the with of correction terms will take the form

$$TSu=\int_c^d\int_a^b f(x,y)\,dxdy=\frac{h^2}{8}[\ f(a,c)+f(a,d)+f(b,c)+f(b,d)+2(f(a+(n-0.5)h,c)$$

$$+f(a+(n-0.5)h,d))+2\sum_{i=1}^{n-1}(f(a,y_i)+f(b,y_i)+2f(a+(n-0.5)h,y_i)+f(x_i,c)$$

$$+f(x_i,d)++f(a+(i-0.5)h,c)+f(a+(i-0.5)h,d)+4\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(f(x_i,y_j)+f(a+(i-0.5)h,y_j))\ ]$$

$$+A_{TSu}h^2+B_{TSu}h^4+C_{TSu}h^6+...$$

where $B_{SuSu},C_{SuSu}\ A_{SuSu}$ constants whose value depends on the partial derivatives of the function $f(x,y)$,and that $i=1,2,3...,n-1$, $x_i=a+ih$, $j=1,2,3,...,n-1$, $y_j=c+jh$

5-Examples:-

1. $\int_1^2\int_1^2 \ln(x+y)\,dxdy$ The analytical value is (1.08913865206603) approximated to 14 decimal places.
2. $\int_3^4\int_0^1 xe^{-(x+y)}\,dxdy$ The analytical value (0.06144772819733) near to fourteen decimal places
3. $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dxdy$ has no analytical value.

6-Results: - For the first integration, its plot appears in Figure (1), the results are display in tables 1,2,3,4 by using *RTT, RSuSu ,RSuT, RTSu* respectively we will obtain following results

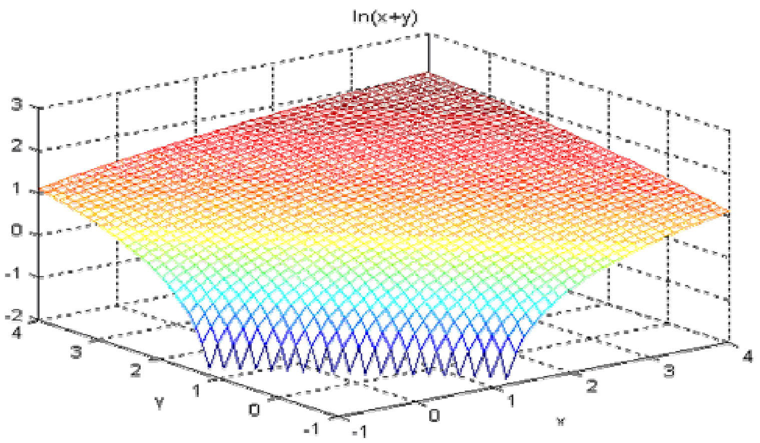

Figure 1:Geometric representation of ln(x+y) within the region of integration and around at (1)

n=m	TT	K=2	K=4	K=6	K=8	K=10
1	1.06916652975401					
2	1.08420812969791	1.08922199634588				
4	1.08791028604728	1.08914433816373	1.08913916095159			
8	1.08883183373270	1.08913901629450	1.08913866150322	1.08913865357547		
16	1.08906196466495	1.08913867497570	1.08913865222111	1.08913865207378	1.08913865206789	
32	1.08911948129137	1.08913865350018	1.08913865206848	1.08913865206606	1.08913865206603	1.08913865206603

Table 1:shows the calculation of double integral $\int_1^2 \int_1^2 \ln(x+y)\,dx\,dy$ using method RTT

n=m	SuSu	K=2	K=4	K=6	K=8
1	1.08420812969791				
2	1.08791028604728	1.08914433816373			
4	1.08883183373270	1.08913901629450	1.08913866150322		
8	1.08906196466495	1.08913867497570	1.08913865222111	1.08913865207378	
16	1.08911948129137	1.08913865350018	1.08913865206848	1.08913865206606	1.08913865206603

Table 2:shows the Double integration calculation $\int_1^2 \int_1^2 \ln(x+y)\,dx\,dy$ to method RSuSu

n=m	SuT	K=2	K=4	K=6	K=8
1	1.07684668996939				
2	1.08606992611514	1.089144433816373			
4	1.08837174374966	1.08913901629450	1.08913866150322		
8	1.08894694216919	1.08913867497570	1.08913865222111	1.08913865207378	
16	1.08909072566744	1.08913865350018	1.08913865206848	1.08913865206606	1.08913865206603

Table 3:shows the Double integration calculation $\int_1^2 \int_1^2 \ln(x+y)\,dx\,dy$ to method RTSu

n=m	TSu	K=2	K=4	K=6	K=8
1	1.07684668996939				
2	1.08606992611514	1.08914433816373			
4	1.08837174374966	1.08913901629450	1.08913866150322		
8	1.08894694216919	1.08913867497570	1.08913865222111	1.08913865207378	
16	1.08909072566744	1.08913865350018	1.08913865206848	1.08913865206606	1.08913865206603

Table 4:shows the Double integration calculation $\int_1^2 \int_1^2 \ln(x+y)\,dx\,dy$ to method RTSu

From table 1, it is observed by Mohammad [4] using TT that when n=m=32, the above integral value is correct to four decimal places, while is equal to the analytical value if Romberg acceleration was used. On the other hand we deduce from table 2 that using *SuSu* for the case when n=m=16 the value of above integral is correct to four decimal places too, while is equal to the analytical values if Romberg acceleration rule together with the mentioned rule are applied. Moreover, the table 3 shows that using *SuT* for n=m=16 gives a correct value for the above integral for three decimal places, but Romberg acceleration rule together with the mentioned yields the same value of analytical one. Finally, we occlude from table 4 for the case n=m=16, the integral value will be correct to three decimal places if *TSu* was applied, while we obtain an identical value to the analytical value if Romberg acceleration rule together with the mentioned one were applied.

From the first column in each one of above four tables which respectively correspond to the methods *TT*, *SuSu*, *SuT*, *TSu* , we deduce that method *SuSu* converges to the real (analytical) values faster in comparison with other methods with less number of sub-intervals. For example, when n=m=16 the value of the integral given by *SuSu* is correct to four decimal places, while in the other methods the resulting value is correct to three decimal places.

For the second integration, its plot appears in Figure (2), the results are display in tables 5, 6, 7, 8 by using *RTT*, *RSuSu* ,*RSuT*, *RTSu* respectively we will obtain following results

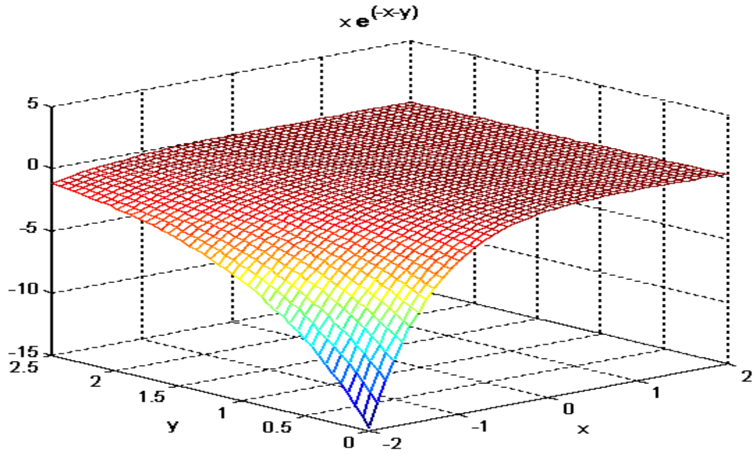

Figure 2:Geometric representation of $xe^{-(x+y)}$ within the region of integration and around at (2)

n=m	TT	K=2	K=4	K=6	K=8
1	1.08420812969791				
2	1.08791028604728	1.08914433816373			
4	1.08883183373270	1.08913901629450	1.08913866150322		
8	1.08906196466495	1.08913867497570	1.08913865222111	1.08913865207378	
16	1.08911948129137	1.08913865350018	1.08913865206848	1.08913865206606	1.08913865206603

Table 5:shows the Double integration calculation $\int_3^4 \int_0^1 xe^{-(x+y)}\,dxdy$ to method RTT

n=m	SuSu	K=2	K=4	K=6	K=8
1	0.05782362963111				
2	0.06055283133530	0.06146256523669			
4	0.06122471007084	0.06144866964935	0.06144774327686		
8	0.06139201796392	0.06144778726162	0.06144772843577	0.06144772820020	
16	0.06143380341025	0.06144773189235	0.06144772820107	0.06144772819734	0.06144772819733

Table 6:show the double integration calculation $\int_3^4 \int_0^1 xe^{-(x+y)}\,dxdy$ to method RSuS

n=m	SuT	K=2	K=4	K=6	K=8	K=10
1	0.04366154360653					
2	0.05694308067686	0.06137025970030				
4	0.06031782705260	0.06144274251118	0.06144757469857			
8	0.06116501747462	0.06144741428196	0.06144772573335	0.06144772813072		
16	0.06137703577462	0.06144770854128	0.06144772815857	0.06144772819707	0.06144772819733	
32	0.06143005416985	0.06144772696826	0.06144772819673	0.06144772819733	0.06144772819733	0.06144772819733

Table 7:shows the double integration calculation $\int_3^4 \int_0^1 xe^{-(x+y)}\,dxdy$ to method RSuT

n=m	TSu	K=2	K=4	K=6	K=8	K=10
1	0.06129218879710					
2	0.06148920027908	0.06155487077308				
4	0.06146324766041	0.06145459678752	0.06144791185515			
8	0.06145193209606	0.06144816024128	0.06144773113820	0.06144772826967		
16	0.06144879945658	0.06144775524343	0.06144772824357	0.06144772819762	0.06144772819734	
32	0.06144799728044	0.06144772988839	0.06144772819806	0.06144772819733	0.06144772819733	0.06144772819733

Table 8:shows the Double integration calculation $\int_3^4 \int_0^1 xe^{-(x+y)}\,dxdy$ to method RTSu

From table 5, it is observed by Mohammad [4] using TT that when n=m=32, the above integral value is correct to four decimal places, while is equal to the analytical value if Romberg acceleration was used. On the other hand we deduce from table 6 that using *SuSu* for the case when n=m=32 the value of above integral is correct to four decimal places too, while is equal to the analytical values if Romberg acceleration rule together with the mentioned rule are applied. Moreover, the table 7 shows that using *TSu* for n=m=32 gives a correct value for the above integral for six decimal places, but Romberg acceleration rule together with the mentioned rule yield the same value of analytical one. Finally, we occlude from table 8 for the case n=m=32, the integral value will be correct to three decimal places if *TSu* was applied, while we obtain an identical value to the analytical value if Romberg acceleration rule together with the mentioned one were applied.

From the first column in each one of the above four tables which respectively correspond to the methods *TT, SuSu, SuT, TSu* , we deduce that method *SuSu* converges to the real (analytical) values faster in comparison with other methods with less number of subintervals. For example, when n=m=16 the value of the integral given by *SuSu* is correct to four decimal places, *SuT* to three decimal places and *TSu* to five decimal places.

For the third integration, its plot appears in Figure (3), the results are display in tables 9, 10, 11, 12 by using *RTT, RSuSu ,RSuT, RTSu* respectively we will obtain following results

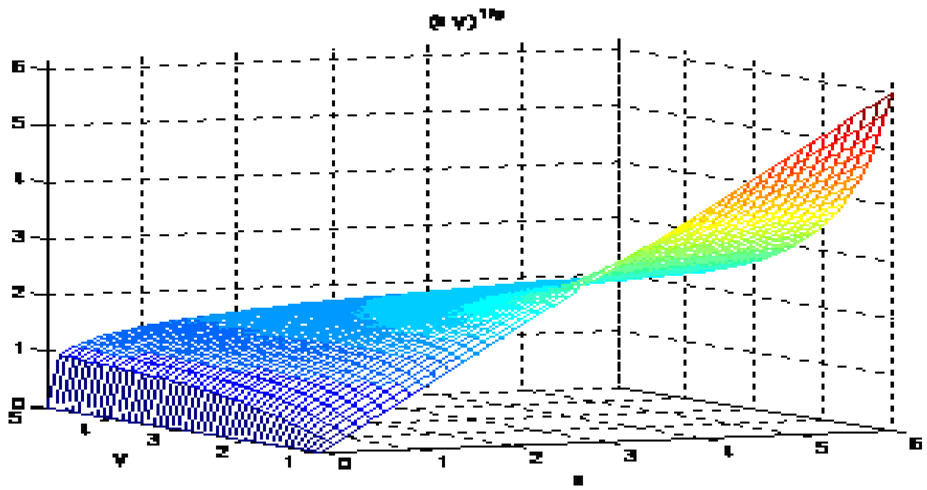

Figure 3:Geometric representation of $(xy)^{\frac{1}{y}}$ within the region of integration and around at (3)

n=m	TT	K=2	K=4	K=6	K=8	K=10
1	2.08667353966681					
2	2.08401453957666	2.08312820621328				
4	2.08339748332247	2.08319179790441	2.08319603735048			
8	2.08324720403810	2.08319711094330	2.08319746514590	2.08319748780931		
16	2.08320990405829	2.08319747073169	2.08319749471759	2.08319749518698	2.08319749521591	
32	2.08320059628184	2.08319749368968	2.08319749522022	2.08319749522819	2.08319749522835	2.08319749522837
64	2.08319827041953	2.08319749513209	2.08319749522825	2.08319749522838	2.08319749522838	2.08319749522838

Table 9:shows the Double integration calculation $\int_2^3 \int_2^3 (xy)^{\frac{1}{y}} \, dx dy$ to method RTT

n=m	SuSu	K=2	K=4	K=6	K=8	K=10
1	2.08401453957666					
2	2.08339748332247	2.08319179790441				
4	2.08324720403810	2.08319711094330	2.08319746514590			
8	2.08320990405829	2.08319747073169	2.08319749471759	2.08319749518698		
16	2.08320059628184	2.08319749368968	2.08319749522022	2.08319749522819	2.08319749522835	
32	2.08319827041953	2.08319749513209	2.08319749522825	2.08319749522838	2.08319749522838	2.08319749522838

Table 10:shows the Double integration calculation $\int_2^3 \int_2^3 (xy)^{\frac{1}{y}} \, dx dy$ to method RSuSu

n=m	SuT	K=2	K=4	K=6	K=8	K=10	K=12
1	2.09171221935446						
2	2.08529079588142	2.08315032139040					
4	2.08371776627863	2.08319342307770	2.08319629652352				
8	2.08332735439569	2.08319721710137	2.08319747003628	2.08319748866347			
16	2.08322994668058	2.08319747744221	2.08319749479827	2.08319749519131	2.08319749521691		
32	2.08320560725286	2.08319749411029	2.08319749522150	2.08319749522822	2.08319749522836	2.08319749522837	
64	2.08319952318201	2.08319749515839	2.08319749522827	2.08319749522838	2.08319749522838	2.08319749522838	2.08319749522838

Table 11:shows the Double integration calculation $\int_2^3 \int_2^3 (xy)^{\frac{1}{y}} dxdy$ to method RSuT

n	TSu	K=2	K=4	K=6	K=8	K=10
1	2.07896212037993					
2	2.08211986237696	2.08317244304264				
4	2.08292682604702	2.08319581393704	2.08319737199666			
8	2.08312974759935	2.08319738811680	2.08319749306211	2.08319749498379		
16	2.08318055327565	2.08319748850108	2.08319749519337	2.08319749522720	2.08319749522815	
32	2.08319325942446	2.08319749480740	2.08319749522782	2.08319749522837	2.08319749522837	2.08319749522838

Table 12:shows the Double integration calculation $\int_2^3 \int_2^3 (xy)^{\frac{1}{y}} dxdy$ to method RTSu

As far as the fourth integral $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dx\,dy$ concern, a plot of which is shown in Figure 3, its analytical value is unknown. Tables 9,10,11 and 12 are respectively results of applying the above four methods. It appear to us that the values are the same horizontally in three successive columns when m=n=32 using *RSuSu* , one column when *RTSu* is applied, and four columns for the other two cases *RSuT* and *RTT*. This means that the actual value of the above integral is (2.08319749522838) rounded to 14 decimal places.

Discussion and conclusion: Three theorems were proved to solve double integrals over their given intervals. From the tables corresponding to the rules *SuT*, *YSu*, *SuSu* and *TT* we conclude that they give good results, but they need relatively high number of sub-intervals. But using Romberg acceleration after external adjustment, we reach better results which were closer to the real values of the integrals. Comparing the four methods left us to deduce that *RSuSu* is the best among the other methods, which gave in all selected examples more accurate and fast convergence to the real value with less number of subintervals.

اشتقاق قواعد مركبة لحساب التكاملات الثنائية عددياً واستخدام تعجيل رومبرك لتحسين النتائج

مقدم من قبل

م.رحاب علي خضير

جامعة الكوفة/كلية التربية للبنات/قسم الرياضيات

المستخلص : الهدف الرئيسي في هذا البحث هو إيجاد قيم التكاملات الثنائية البعد عددياً التي مكاملاتها مستمرة, وذلك من خلال اشتقاق قواعد مركبة وإيجاد حدود التصحيح لها , وبالاعتماد على هذه الحدود قمنا بتعجيل تقارب القيم العددية باستخدام تعجيل رومبرك لتحسين النتائج, ثم استخدمنا هذه الطرائق لإيجاد قيم التكاملات ذات المكاملات المستمرة في مناطق تكاملها وقارنا بين النتائج التي حصلنا عليها من تطبيق هذه القواعد مع النتائج التي استخدمت فيها طريقة شبه المنحرف على البعدين الخارجي والداخلي, فكانت القواعد التي تم اشتقاقها أفضل من حيث الدقة وسرعة الاقتراب إلى القيم التحليلية (الحقيقية) بأقل عدد من الفترات الجزئية.

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